1-1-2005

Empirical tests of parametric and non-parametric Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR) measures for the Brazilian stock market index

Luciano Martin Rostagno
Iowa State University

Empirical tests of parametric and non-parametric Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR) measures for the Brazilian stock market index

by

Luciano Martin Rostagno

A thesis submitted to the graduate faculty in partial

fulfillment of the requirements for the degree of

MASTER OF SCIENCE

Major: Economics

Program of Study Committee:
P. Marcelo Oviedo, Major Professor
Dermot J. Hayes
Gary D. Koppenhaver

Iowa State University

Ames, Iowa

2005

Graduate College

Iowa State University

This is to certify that the master's thesis of

Luciano Martin Rostagno

has met the thesis requirements of Iowa State University

Signatures have been redacted for privacy

Table of contents

List of tables

List of figures

Abstract

This study aims to verify empirically the accuracy of parametric and non-parametric approaches in estimating Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR) measures of the Brazilian stock market index (Ibovespa). The period of analysis goes from the first day of trade of 1995 to the last day of trade of 2004, which is used for estimation and test of the risk parameters. Parametric approaches assume that daily returns follow a normal and a t-distribution. Non-parametric approaches are the historical simulation and the volatility-weighted historical simulation technique. The binomial test is applied to verify if the failure rates predicted by VaR measures given by the models are acceptable and the sample differences paired test is used to evaluate the accuracy of the CVaR measures in forecasting tail losses. The results point out that the volatility-weighted historical simulation approach gives better estimates of both measures of risk. The rates of losses exceeding volatility-weighted historical simulation VaRs (VWHS-VaRs) ranged between 4.7-6.0%, at the 95% *cl*, and between 0.9-1.2%, at the 99% *cl*. For all periods of estimation used (1, 2, 3, 4, and 5 years), at the 95% *cl*, the sample differences paired test indicated no statistically significant differences between the VWHS-CVaR estimates and the losses beyond its VaR estimates. Risk lines for the normal and historical simulation VaR (HS-VaR) estimates presented flatness, or excessive smoothness, for large periods of estimation, and the student *t* VaR (T-VaR) estimates were sometimes too low or too high. For these models, short periods of estimation gave more accurate VaR estimates. For the CVaR estimates, the normal and *t*-distribution assumptions caused overestimation of the value of the tail losses. Finally, the HS-CVaR had similar performance of HS-VaR providing, at the 95% *cl*, good estimates of tail losses when short periods of estimation were used.

1. Introduction

The interest on the issue of risk management has considerably increased in the past decades. Volatile economic environment, growth in trading activity, and advances in information technology are among the factors that contributed to increase the interest and development of risk management tools.[1]

Markowitz (1952) was the first to explicitly include risk analysis in a portfolio selection setting.[2] The risk, in this case, was measured in terms of variances and covariances between securities. Basically, Markowitz proposed a combination of statistical analyses to estimate means, variances, and covariances of securities that, when included in a portfolio, define the relation between risk and return. The author called the optimal combinations of securities, which lead to the maximum return possible for each given risk level, as the set of mean-variance efficient combinations.

Based on the portfolio theory developed by Markowitz, some economic models of asset pricing also used risk analysis to determine differences in asset returns. Essentially, these models seek to quantify risk through risk measures and then convert a random future gain or loss into a certainty equivalent.

The first well-known economic model of asset pricing is the CAPM (Capital Asset Pricing Model). Presented by Sharpe (1964), Lintner (1965), and Mossin (1966), this model suggests that the market beta is the only factor defining expected returns. In simple terms, the market beta expresses the sensitivity of an asset relative to the market. Statistically, it is defined as the covariance between an asset returns and the market returns divided by the variance of the market returns.

---

[1] Dowd (2002, Ch. 1)

[2] "Markowitz proposed to measure the risk associated to the return of each investment by means of the deviation from the mean of the return distribution, the variance, and in the case of a combination (portfolio) of assets, to gauge the risk level via the covariance between all pairs of investments" (Szegö, 2002).

Later, seeking to overcome the weaknesses and limitations of the CAPM evidenced by other studies (Black (1972) and Merton (1973)), Ross (1976) developed an alternative model denominated Arbitrage Pricing Theory (APT). The idea behind this model was to construct a multiple factor model incorporating alternative sources of risk in an economy not included in the CAPM. Its basic assumption is that two portfolios with the same risk level should not have different expected returns. On the contrary, the arbitrage process would promptly eliminate the difference. The model assumes that the sensitivity of an asset price to some factors determine the risk premium. Expected returns and factors (betas) are linearly related. The theory does not specify which factors should be used in the model, but suggests that they are macroeconomic factors responsible for the systematic risk, the risk that cannot be eliminated through the diversification process.

In summary, the main economic asset pricing models (CAPM and APT) use exogenous information to explain expected returns. Variables such as market risk, industrial production, changes in the risk premium, twists in the yield curve, measures of unanticipated inflation and changes in expected inflation are found to influence on pricing (Chen, Roll, and Ross (1986)). The risk of an asset is, therefore, estimated and the individual expected return is specified. On this kind of approach, the relative risk is the determinant factor of the differential asset returns.

The importance of Markowitz's work for risk management in firms and banks is also notorious. Essentially, his study raised a new issue that had never been addressed before. After his work, researches and financial institutions have been seeking to develop risk measurement tools to deal with the exposure to adversity. Different models and approaches to measure risk were created to better fit to institutions needs.

In this context we have gap analysis, whose purpose was to give an idea of interest-rate risk exposure faced by financial institutions. In simple terms, the gap analysis measures the change in net interest income due to change in interest rates. It is determined by the difference between repricing assets and repricing liabilities. Therefore, a negative gap indicates a risk exposure to increasing rates. As pointed by Dowd (2002, p.4), although simple to carry out the gap analysis has its own limitations: "it only applies to on-balance sheet interest-rate risk, and even then only crudely; it looks at the impact of interest rates on income, rather than on asset or liability values; and results can be sensitive to the choice of horizon period."

Another method developed to measure interest-rate risk was duration analysis.[3] It captures the sensitivity of an asset price to movements in yields. The advantage of this approach over the gap analysis is that it focuses on changes in asset (or liability) values, rather than just change in net income. However, the method has also some pitfalls: it ignores risks other than interest-rate risk; it only takes a first-order approximation to the change in the bond price;[4] and it only considers parallel shifts in interest rates.[5] As Lam (2003, p.184) states, "in real life, shifts in the interest rate curve are often anything but parallel." To circumvent the parallel hypothesis other measures of duration have been created. Denominated by Ho (1992) as *key rate durations*, these powerful tools have become popular in measuring non-parallel risks. The author defines them as a vector representing the price sensitivity of a security to each key rate change. The main attribute of these tools is that they assume that multiple market factors drive the yield curve movement. Like the others, however, the key rate durations approaches have also some shortcomings: they are unintuitive requiring some experience and familiarization; they ignore correlations between shifts at different reference maturities; and they are based on perturbing a theoretical zero coupon curve rather than observed yields on coupon bonds, which introduces some arbitrariness into the results.[6]

A different approach to measure risk is the scenario analysis. In this technique possible scenarios (product of an event or combinations of events) are created to quantify its impact

---

[3] Mathematically, duration is defined as follows: $D = \sum_{i=1}^{n} [i \times PVCF_i] \Big/ \sum_{i=1}^{n} PVCF_i$, where $PVCF_i$ is the present value of the period i cash flow, discounted at the appropriate spot period yield.

[4] Dowd (2002, p.4)

[5] Lam (2003, p.184)

[6] For more details see Phoa (2001).

on the enterprise. The purpose is to determine the size of potential losses related to specific scenario. As nicely posted by Lam (2003, p.192):

> **"Scenario analysis typically goes beyond the immediate effects of predefined market moves and tries to draw out the broader impact that events may have on the revenue stream and business. It is meant to help management understand the impact of unlikely but catastrophic events, such as major changes in the external macroeconomic environment that will have an effect well beyond any immediate impact on the value of a trading portfolio."**

In a few words, the scenario analysis works as follows. Once defined the relevant variables and how they evolve over time, the risk manager analyses the risk exposure by looking at the possible results for each scenario assumed. An important aspect of this approach is that it relies on subjective choices made by the analyst. Further, it tells nothing about the probability of occurrence of each scenario. So depending on the choices and beliefs of the analyst, the scenario analysis can bring diverse results and, consequently, lead to different actions.[7]

More recently, in the mid-90s, the bank JP Morgan developed a standard risk measure for financial risk management, called Value-at-risk (henceforth VaR). The idea was to create a general measure of economic loss that could incorporate the diverse risk across positions and aggregate them on a portfolio basis. In simple terms, VaR gives the maximum loss possible over a period of time at a predefined confidence level. It is essentially a static model (one-period model) like scenario analysis. Positions remain unchanged over the risk horizon.

In fact, there are three different approaches to the calculation of VaR: parametric VaR, Monte Carlo simulation, and historical simulation. All forecast risk by analyzing historical patterns of market variables but they also differ in some aspects. Importantly, these methods are not exclusives and can be used together to provide a more robust VaR estimates.[8]

---

[7] Dowd (2002, p.5)
[8] Gallati (2003)

The most common and also the simplest one is the parametric VaR. Fast and simple to calculate, this method requires only the parameters defining the distribution of the historical data. Having this information, the analyst can easily estimate the VaR parameter by fitting a curve through the data. However, this approach assumes that the changes in the portfolio value follow a certain distribution, implying that the method is accurate when the distribution choices are correct. Therefore, when using this method the analyst should carefully consider the statistical distribution and the data used.

Nevertheless, the traditional models of VaR suffer from some shortcomings. One critical aspect of the analysis using parametric VaR relates to the assumptions on the distributional properties of the underlying risk factors. Violations of the assumptions can lead to misleading results. For example, when the distribution of losses is not normal, which is generally the case (Hendricks, 1996), the model doesn't perform well when the normality assumption is assumed.

In the historical simulation approach, returns are obtained from the time series of historical asset prices. It is essentially a nonparametric method, being independent regarding nonlinearities and nonnormal distributions. So, the returns of the risk factors do not need to be independent and normally distributed over time as in the parametric method. Because it is based on actual prices, this approach considers fat tails. As long as extreme events are contained in the data set, they are considered in the simulation. On the other hand, this method also has some criticisms. The model is based on the past to foretell the future. This brings extreme dependence on a particular historical data set and its idiosyncrasies. Also, the model cannot accommodate changes in market structure and may not adequately represent current market conditions. Finally, the quality of its estimation is linked to the length of the data set.[9]

[9] For a more detailed discussion about the VaR approaches see Gallati (2003, Ch. 5) and Crouhy, Galai and Mark (2001, Ch. 4).

The Monte Carlo simulation approach follows the same idea behind the historical simulation method, except that it generates stochastic returns using a defined stochastic process and statistical parameters that drive the process. In this case, the distributions of the risk factors are generated by using a model created from a set of random variables. The special feature of this kind of approach is that it can be applied to any analytic multivariate distribution for the risk factors. Distributions with fat tails and skewness (asymmetry), like student *t*-distributions, are among the possible distributions in which the analyst can adopt to calculate the VaR using Monte Carlo simulation.

Comparing the Monte Carlo simulation approach and the historical simulation technique, it can be noted that the former is more computationally intensive than the last one. It involves re-valuating the portfolio under each scenario and, as pointed by Hawkins (2000, p.146), it requires a large number of simulations or paths before the VaR converges towards a single number.

A general shortcoming of VaR approaches is that different models describing the market behavior can result in different outputs. For instance, Beder (1995) shows that different results for same portfolios can occur depending on which VaR technique is adopted. Furthermore, the results reveal that the VaR measure is highly dependent on parameters, data, and assumptions.

But perhaps the most serious conceptual problem with VaR is that it disregards any loss beyond its critical value. This problem, called the tail risk, was address by Artzner et al. (1997) and Embrechts et al. (1998). Being aware of this fact is so critical that, as shown by Yamai and Yoshiba (2005), in certain real-world cases investors may take wrong decisions based on VaR. The example presented by the authors is an investor who assembles his portfolio without taking into account the magnitude of unlikely losses (losses beyond the VaR critical value) that eventually may occur. The authors also show that in high volatile markets or in markets in which the assets have extreme dependence structure, VaR may underestimate risk. Further, using a dynamic portfolio optimization framework, Basak and

Shapiro (2001) show that an investor who wants to maximize his utility under VaR constraint could result in choosing a position that can lead to losses beyond the VaR critical value. Additionally, Yamai and Yoshiba (2002) present the same problem for the case of a concentrated credit portfolio and far-out-of-the-money option.

To circumvent the main weakness of the VaR technique, Rockafellar and Uryasev (2000, 2002) suggest the Conditional Value-at-Risk, or CVaR,[10] which takes into account losses outside the VaR quantile. Supported by the extreme value theory (EVT), the CVaR represents the conditional expectation of loss (mean loss) that is beyond the VaR level. It basically indicates what we can expect to lose given the loss is beyond the VaR. Consequently, the CVaR will always produce a risk measure that exceeds the VaR. As pointed by Dowd (2002, Ch. 2), the CVaR has the many attractions of the VaR measure and, further, it has also some additional advantages: the CVaR tells the analyst what to expect in bad states (tail events); it does not discourage risk diversification as the VaR sometimes does; and the CVaR estimates are less prone to sampling error than VaR.

Finally, there is the stress testing approach. It is essentially a scenario analysis but totally focused on crisis situations. The main objective of this kind of analysis is to capture the vulnerability of a portfolio or position to hypothetical events. It seeks to quantify the magnitude of loss if abnormal situations eventually occur. As pointed by Dowd (2002, p.202) the stress testing should be used as a complement to probability-based risk measures such as VaR and CVaR because it gives a lot of information about bad states, which is missed by the probabilistic approaches.

This study aims to verify empirically the applicability of parametric and non-parametric approaches to measure risk (VaR and CVaR) for the Brazilian market index (Ibovespa) traded on the Sao Paulo Stock Exchange. The period used for analysis goes from the first day of trade of 1995 to the last day of trade of 2004. Parametric approaches used assume that

[10] This risk measure is also called in the literature as expected tail loss, expected shortfall, tail VaR, tail conditional expectation and worst conditional expectation.

daily returns follow a normal and a t-distribution. Non-parametric approaches adopted are the historical simulation and the volatility-weighted historical simulation technique.

The remainder of this work is structured as follows. In the next section, formal definitions of VaR and CVaR measures are presented. Section 3 introduces the methodology used in this study. Section 4 reports the results of the tests applied. Section 5 briefly summarizes our findings.

2. VaR and CVaR: formal definitions

Since Value-at-Risk (VaR) was developed11, it has become a standard tool of risk measure among financial institutions. The idea behind the technique is that, given a time horizon and confidence level, VaR estimates the maximum loss that can possibly occur. The appeal of the VaR measure relies on the fact that it aggregates the several components of risk into one single number. Additionally, as pointed out by Penza and Bansal (2001, p.62), the technique is flexible in the sense that by only choosing the appropriate period of time and probability level, the tool can be adapted to institutions' specific needs.

As an alternative measure, the Conditional Value-at-Risk (CVaR) came to overcome some weaknesses found for the VaR measure. It is essentially the expected value of the losses beyond VaR. The CVaR measure is directly related to the VaR measure.

There are different ways to measure VaR and, consequently, CVaR.12 They can be classified in two broad categories: parametric approaches and non-parametric approaches. This section presents the formal definitions of the approaches used in this study to measure VaR and CVaR.

2.1 Parametric approaches and VaR

The parametric approaches seek to estimate VaR and CVaR measures based on assumptions about the distribution of returns. In this case, it is necessary to explicitly specify the distribution from which the data observations are drawn. The distribution assumptions assumed to estimate VaR and CVaR from the parametric approaches used in this study are the normal and the student t-distribution. Because daily stock returns are found to have fat tailed distribution (Campbell et al. (1997)), it is expected that the VaR and CVaR measures

[11] See JP Morgan (1994, 1997) and Phelan (1997).
[12] See Duffie and Pan (1997) for an overview.

from the *t*-distribution will be more accurate than when assumed that the returns are normally distributed.

The general form of the VaR measure, which is valid to any distribution (discrete or continuous), can be obtained from the probability distribution of returns f (R). In this case, we find the worst possible realization R* (the cut-off return) for a given confidence level (*cl*), such that the probability of a value lower than R^*, $p = P(R \leq R^*)$, is 1-*cl*:[13]

$$1 - cl = \int_{-\infty}^{R^*} f(R)dR = P(R \leq R^*) = p \tag{1}$$

One assumption that can be made when using a parametric approach is that returns are normally distributed. So, if the returns of a position, *R*, are normally distributed with mean, μ_R, and standard deviation, σ_R, then the analytic density function is given by:[14]

$$f(R) = \frac{1}{\sigma_R \sqrt{2\pi}} \exp\left[-\frac{1}{2} \left((R - \mu_R)/\sigma_R \right)^2 \right] \tag{2}$$

If *cl* is the confidence level and R^* is the cut-off return, then follows that the VaR in terms of returns is:

$$VaR^R = R^*_{cl} = \mu_R + \alpha_{cl}\sigma_R \tag{3}$$

The parameter α_{cl} represents the standard normal variate corresponding to the confidence level *cl* (e.g., $\alpha_{0.95} = -1.645$ and $\alpha_{0.99} = -2.326$). It can be directly obtained from standard statistical tables.

---

[13] Jorion (2001, p.110)

[14] This derivation follows the one presented by Grouhy, Galai and Mark (2001, p.192).

Graphically, we can represent the VaR estimation as follows. If, for example, we choose a holding period t (it might be one day, one week or one month, for example), which is the period of time over which the parameter of interest (for example, profit/loss or asset return) is measured, and a confidence level cl of 95 percent, the Value-at-Risk is the loss in market value over the period of estimation that is exceeded with probability 1-cl. Supposing that the data we are interested in analyzing is normally distributed with mean 0 and standard deviation 1 over the holding period t, the VaR estimate at the 95% cl is -1.645 and at 99% cl is –2.326. Figure 1 illustrates both cases.

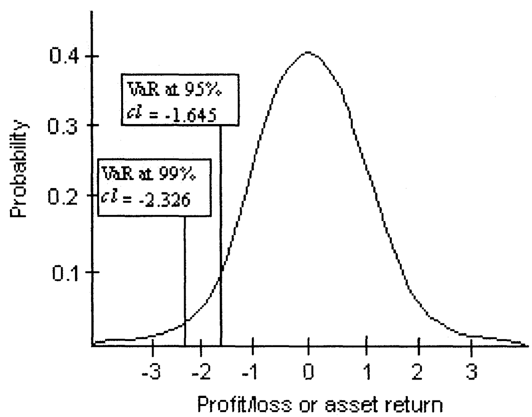

Figure 1: VaRs at the 95% and 99% confidence levels from a standard normal distribution.

The probability density function shows the point on the x-axis corresponding to the VaR, which is the critical value or maximum loss for the confidence level chosen. It is easy to note that the VaR is increasing in confidence level and, in general, in longer holding periods. As pointed by Jorion (2001, p.116), under certain conditions one can obtain the same value of VaR increasing one or the other factor.

For arithmetic returns, R^A, assuming that all dividends are reinvested continually in the asset itself, the asset return is given by the formula:

$$R_t^A = \frac{(P_t - P_{t-1})}{P_{t-1}} \tag{4}$$

Where P represents the asset price in different times. Associating the VaR measure in terms of arithmetic returns, VaR^{R^A}, with the VaR measure in terms of the critical value of price (P^*), VaR, follows that:

$$R^{A^*} = \frac{(P^* - P_{t-1})}{P_{t-1}} = \frac{VaR}{P_{t-1}} \tag{5}$$

And therefore:

$$VaR = (\mu_{R^A} + \alpha_{cl}\sigma_{R^A})P_{t-1} = VaR^{R^A} \times P_{t-1} \tag{6}$$

As we can see, it is easy to obtain the VaR measure in terms of losses in price from the VaR measure in terms of arithmetic returns and vice-versa. However, when calculating one of these VaRs is important that the distribution assumption be coherent with the data. In this case, the assumption may vary depending the type of data used (profit/loss or return).

2.1.1 VaR and distribution properties

An important aspect of the distribution assumption is the data fit. Violations on the distribution assumption should be considered carefully; otherwise the outcomes could lead to major errors in the risk analysis. So, before extracting conclusions, one should first find the distribution that better represents the data.

For that we can check if the distribution generated by the data approximates to the normal or some other type of distribution. The normal distribution is fully described by two parameters, mean and standard deviation. Further, it is characterized by been symmetric (zero-skew) with a kurtosis of 3.

To verify the asymmetry or skewness of the distribution, the third moment is the appropriate measure. Mathematically, it is calculated as follows:

$$Skew = E(R-\mu)^3/\sigma^3 \tag{7}$$

A zero skewness coefficient means that the distribution is symmetric. A non-zero and positive skewness coefficient means that the distribution is not symmetric and has a long tail on the right and a short tail in left side. A negative number indicates the opposite. Therefore, the sign of the skewness coefficient indicates the direction of the skew.

An example of a non-symmetric distribution is the lognormal distribution. Because the price of an asset (or portfolio) cannot be negative, the resulting distribution has a short tail in the left and a long tail in the right side. Figure 2 illustrates this case.

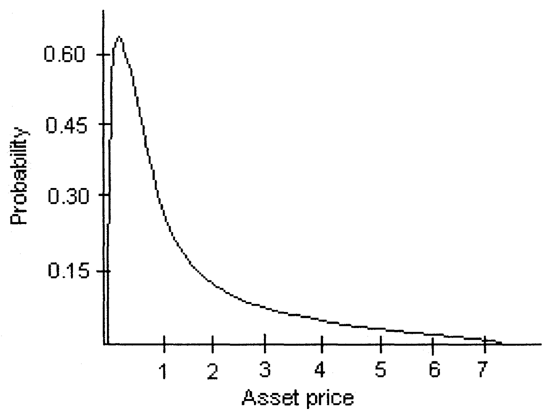

Figure 2: A lognormally distributed asset price

In terms of returns, considering the aspect of non-negative prices, working with geometric returns is more plausible than with arithmetic returns. Assuming that all dividends are reinvested continually in the asset itself, the geometric return, R^G, is given by the formula:

$$R_t^G = \log\left(\frac{P_t}{P_{t-1}}\right) \tag{8}$$

In this case, the assumption of normally distributed returns becomes more realistic because the left tail of the distribution such as $\log(P_t / P_{t-1}) \to -\infty$ is achieved as $(P_t / P_{t-1}) \to 0$ or $P_t \to 0$. Therefore, the assumption that geometric returns are normally distributed implies that the logarithm of P_t follows a normal distribution and P_t itself follows a lognormal distribution. For arithmetic returns, the assumption of normal distribution implies that the left tail $(P_t - P_{t-1})/P_{t-1} \to -\infty$ is achieved as $(P_t / P_{t-1}) - 1 < -1$ or $P_t < 0$ which is economically meaningless.[15]

Analogous to the case of arithmetic returns, given a confidence level *cl*, the critical value of a geometric return R^G, R^{G^*} or VaR^{R^G}, is:

$$VaR^{R^G} = R^{G^*} = \mu_{R^G} + \alpha_{cl}\sigma_{R^G} \tag{9}$$

To find the VaR measure in terms of the asset price, first we should find the critical value P^*. This measure is derived directly from the geometric return formula.

$$R_t^{G^*} = \log\left(\frac{P_t^*}{P_{t-1}}\right) = \log P_t^* - \log P_{t-1} \tag{10}$$

$$\Rightarrow \log P_t^* = R_t^{G^*} + \log P_{t-1} \tag{11}$$

[15] Jorion (2001, p.99-100)

$$\Rightarrow P_t^* = \exp\left[R_t^{G^*} + \log P_{t-1}\right] \tag{12}$$

$$\Rightarrow P_t^* = \exp\left[\mu_{R^G} + \alpha_{cl}\sigma_{R^G} + \log P_{t-1}\right] \tag{13}$$

Finally, we can define an expression for the VaR measure for geometric returns in terms of the asset price change. By definition, the VaR is:

$$VaR = P_{t-1} - P_t^* \tag{14}$$

$$\Rightarrow VaR = P_{t-1} - \exp\left[\mu_{R^G} + \alpha_{cl}\sigma_{R^G} + \log P_{t-1}\right] \tag{15}$$

So, due to the asymmetric aspect of asset prices, when considering obtaining the VaR measure in terms of the asset price change from VaR in terms of returns, it is more plausible to use equation 15 rather than equation 6. It rules out the possibility of negative asset prices.

An additional measure that can be used to verify if the distribution assumed fits the data is the kurtosis parameter. It is measured by the fourth moment of the distribution. The kurtosis of a distribution is given by the following equation:

$$Kurtosis = E(x-\mu)^4 / \sigma^4 \tag{16}$$

This parameter gives a measure of tail flatness of the distribution. For normal distributions the kurtosis is 3. A kurtosis parameter greater than 3 means that the tail of the distribution is fatter than the tail of the normal distribution. A kurtosis parameter less than 3 means the opposite (thinner tails).

As shown by Campbell et al. (1997, ch.1), when analyzing individual stock returns distributions one can expect that the distributions will most likely exhibit "fat tails" (kurtosis > 3). These distributions differ from normal distributions in the sense that they present more observations in the extreme areas.[16] To deal with "fat tails" return distributions, one can use the student *t*-distributions. These distributions are fully described by the mean, μ, and the standard deviation, σ, of the portfolio return, and, additionally, by a parameter controlling the fatness of the tail (the degree of leptokurtosis) denominated "degree of freedom", v. Higher values of v indicates more approximation of the distribution to a normal distribution with similar parameters.[17]

Calculating VaR from a student *t*-distribution is similar to a normal distribution, except that we have to use the appropriate analytic density function $f(R)$.

$$f(R) = \frac{\Gamma[(v+1)/2]}{\Gamma(v/2)} \frac{1}{\sqrt{v\pi}} \frac{1}{(1+R^2/v)^{(v+1)/2}} \tag{17}$$

Where Γ is the gamma function defined as $\Gamma(v) = \int_0^\infty R^{v-1} e^{-R} dx$.

So the VaR measure from a *t*-distribution becomes:

$$VaR^R = R_{cl}^* = \mu_R + \alpha_{cl,v} \sqrt{(v-2)/v}\sigma_R \tag{18}$$

[16] It is important to note, however, that by the central limit theorem, a well-diversified portfolio return might still exhibit a normal distribution even when individual asset returns composing the portfolio do not follow a normal distribution.

[17] Grouhy, Galai and Mark (2001, Ch. 3)

Note that the t VaR formula contains the additional multiplier term $\sqrt{(v-2)/v}$, which moderates the effect of the standard deviation on the VaR. Also, the confidence level term, $\alpha_{cl,v}$, now refers to a student t-distribution instead of a normal one, and so depends on v as well as cl.[18]

As stated by Grouhy, Galai and Mark (2001, p.192), these distributions are of great attention to risk managers because they are aware that extraordinary losses occur more frequently than a normal distribution predicts. Intuitively, the VaR measure obtained from a student t-distribution will always be greater or equal to the VaR measure obtained from a normal distribution.

2.2 Coherent risk measure

An important aspect of a risk measure like VaR is that it must satisfy the characteristics of a coherent risk measure. According to Artzner et al. (1997, 1999) any risk measure should establish a correspondence ρ between the space X of random variables and a non-negative real number; ρ : X → R, in order to be a coherent risk measure.19 Scalar measures of risk allow to order and compare positions according to their respective risk value. Therefore, ρ must satisfy the following properties:

(a) Positive homogeneity: $\rho(\lambda x) = \lambda\, \rho(x)$ for all random variables x and all positive real numbers λ.

(b) Subadditivity: $\rho(x + y) \leq \rho(x) + \rho(y)$ for all random variables x and y; it is easy to note that any positively homogeneous functional ρ, is convex if and only it is subadditive.

(c) Monotonicity: $x \leq y$ implies $\rho(x) \leq \rho(y)$ for all random variables x and y.

---

[18] Dowd (2002, p.83)

[19] For a more detailed discussion about this theory see Frittelli and Gianin (2002) and Szegö (2002).

(d) Transitional invariance: $\rho(x + \alpha r_0) = \rho(x) - \alpha$ for all random variables x and real numbers α, and all riskless rates r_0.

However, VaR turns out to be an inadequate risk measure for the case in which the joint distribution of return is non-elliptical. In this case, VaR may not satisfy the subadditive property behaving as follows:

$$VaR_{cl}(R_1 + R_2) > VaR_{cl}(R_1) + VaR_{cl}(R_2) \tag{19}$$

where R1 and R2 denote the returns of two portfolios and cl denotes the confidence level. This means that the VaR of a combined position may be greater than the sum of the VaRs of the positions considered individually.

Nevertheless, Artzner et al. (1999) show that the quantile-based VaR measure only satisfies the subadditivity property when the return distributions are normal (or more generally, elliptical).

2.3 Parametric approaches and CVaR

To circumvent the problem of non-subadditivity property, Rockafellar and Uryasev (2000, 2002) suggest a different risk measure, called Conditional VaR (CVaR). The CVaR parameter measures the magnitude of potential losses in the tail by determining the expected extreme loss with a predefined confidence level. It is the conditional expectation of loss given that the loss is beyond the VaR level. Mathematically, the Conditional VaR can be defined as follows:

$$CVaR^R = E[R \mid R \leq VaR^R] = \frac{\int_{-\infty}^{R^*} Rf(R)dR}{\int_{-\infty}^{R^*} f(R)dR} \tag{20}$$

The CVaR for the case of a normal distribution is shown in Figure 3. Holding the assumption of standard normal distribution, the CVaR at the 95% confidence level is -2.061.

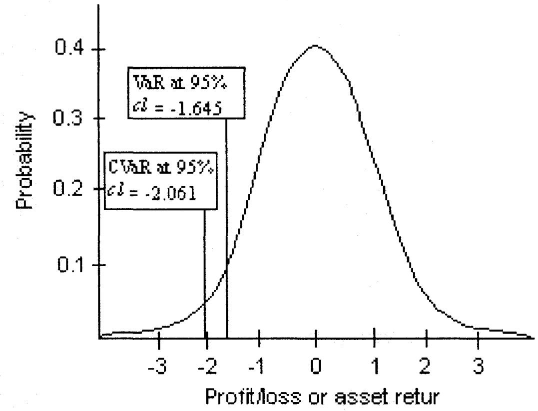

Figure 3: CVaR at the 95% confidence level from a standard normal distribution.

The CVaR is calculated as the average of the tail VaRs at a chosen confidence level. It is the probability-weighted average of tail losses, or losses exceeding VaR. The CVaR measure satisfies all proprieties of a coherent risk measure. So, now aggregating individual risks does not increase overall risk. However, estimating CVaR is computationally more intensive and sometimes its estimation can be significantly more difficult.

2.4 Non-parametric approaches and VaR and CVaR

Non-parametric approaches do not require making assumptions about the distribution of returns. VaR and CVaR measures are obtained directly from the empirical distribution. All non-parametric approaches assume that the near future risk can be forecasted by using data from a precedent period. The most popular non-parametric approach is the historical simulation.[20]

The estimation of VaR using the historical simulation technique consists simply in taking the observation from the frequency histogram drawn from the empirical data. The

[20] There are others non-parametric approaches used to estimate VaR and CVaR as bootstrap methods, non-parametric density estimation methods, and principal components and factor analysis methods.

observation that cuts off the lower one minus the confidence level (i.e.,1-*cl*) of very high losses from the rest of the distribution determines the VaR measure. The CVaR is obtained by taking the average of the tail VaRs.

One important aspect of this kind of approach is the choice of the length of the data period used to estimate the VaR and CVaR measures. Choosing large periods for estimation gives raise the problem of aged data. On the other hand, choosing small periods may lead to imprecise VaR estimations and, sometimes, the CVaR estimate cannot be achieved. Also, if the past data is equally weighted, unlikely to recur past events will influence the estimation of VaR until they get old enough to be excluded. This fact, called ghost effects, causes the estimation being less responsive to current market conditions.

One way to reduce the ghost effects is to modify the simple historical simulation method by given weights to the past data (Boudoukh, et al. (1998)). Age weighting treats most recent observations as having more important information to forecast the near future risk. However, as pointed by Hull and White (1998a), this method reduces the effective sample size. In addition, Pritsker (2001) shows that VaR estimates using age weighting can still be insufficiently responsive to changes in the underlying risk.

An alternative way to weight the data, suggested by Hull and White (1998a), is to use volatility. Volatility-weighted historical simulation takes account recent changes in volatility to update return information. This approach estimates VaR by replacing the returns in the data set, $r_{T,i}$, with volatility-adjusted returns, $r^*_{T,i}$. The VaR and CVaR measures are then found in the usual way. The volatility-adjusted returns are calculated as follows:

$$r^*_{T,i} = \frac{\sigma_{T,i} \cdot r_{T,i}}{\sigma_{t,i}} \tag{21}$$

Where $r_{T,i}$ is the historical return on asset i on day T in the historical sample, $\sigma_{t,i}$ is a historical forecast measure of the volatility of the return on asset i on day t, made at the end of day t-1, and $\sigma_{T,i}$ is the most recent forecast of the volatility of asset i. In this formula, if the current forecast of volatility is greater than the estimated volatility of period t, then the actual returns in any period T will be higher.

To apply this method it is therefore necessary to forecast the volatility of the return on the asset. For that, one can use the exponentially weighted moving average (EWMA) model. The estimates of the volatility using the EWMA model follow the equation:[21]

$$\sigma_t^2 \approx \lambda\sigma_{t-1}^2 + (1-\lambda)r_{t-1}^2 \tag{22}$$

Where the estimate, σ_t, of the volatility for day t (made at the end of day t-1) is obtained from σ_{t-1} (the estimate from one day ago of the volatility for day t-1) and r_{t-1} (the most recent observation on changes in the daily return).

As can be noted the volatility-weighted historical simulation produces risk estimates that are sensitive to volatility changes. Further, it can produce VaR and CVaR estimates greater than the maximum loss in the data set in high volatile periods. So this eliminates limit in VaR and CVaR estimates imposed by standard historical simulation approaches in which the maximum future loss cannot exceed the maximum past loss. Finally, Hull and White (1998a) present empirical evidences that the volatility-weighted historical simulation approach produces superior VaR estimates than the approaches that do not take account of volatility changes.

[21] The derivation of the equation characterizing the EWMA model is presented in Appendix A.

3. Method

3.1 Sample

The sample adopted involves the daily historical returns of the Brazilian stock index[22] (Ibovespa) traded on the Sao Paulo Stock Exchange. The sample starts from the first day of trade of 1995 to the last day of trade of 2004. Figure 4 presents the time series plot of the Brazilian market index values for this period. Numbers are referred to the close value of the day.

Figure 4: Daily quotes of the Brazilian stock index (Ibovespa) from January of 1995 to December of 2004.

From the graph above, it can be noted that there is an overall positive trend in returns. In the first day of trade of 1995, the market was set at 4,319.07 points and in the end of 2004 it reached a total of 26,196.25 points, representing an impressive 506.52% increase. However, there are different moments during the evolution of this market index. Specifically, one can

---

[22] The stocks compounding the Ibovespa index are presented in Appendix B.

observe alternations between positive and negative moments in this market. First, there is a short period in which the market dropped to 2,138.27 points (03/09/1995). Then, the market clearly followed a positive path reaching the peak of 13,617.30 points (07/01/1997). After that, a down period that takes the market back to the 5,000s points is followed by a rapidly recovery during the year of 1999. Next, the market takes a long period of down tendency, accentuated with the approximation of the presidential election when Luis Ignacio Lula da Silva (Lula) was elected. After the uncertainty about the economy was solved and no turn around policies were announced, an expressive positive path followed and the market ended up with the maximum points reached by the index in the whole period.

The choice of the period for the risk analysis (Jan/1995 to Dec/2004) relies on the fact that it is characterized by low inflation rates, which occurred during the Real plan. As Leal and Rego (1997) point out, the high inflation rates observed before the Real Plan distorted asset prices traded on the Brazilian Stock Market (Bovespa). Therefore, the total length of time for estimation and test of each model corresponds to 10 years. When the length of time used for estimation of the first daily VaR is 1 year, the length of time for testing is 9 years. Two years for estimation implies 8 years for testing, and so on.

3.2 Estimation approaches

In this study, parametric and non-parametric approaches (historical simulations) are used to evaluate the effectiveness of different risk measure techniques in the context of daily stock index returns. The accuracy of the VaR estimates is tested by the failure rates predicted by the models and the CVaR measures are compared to the losses exceeding VaR through the sample differences paired test. This section present the procedure used to test empirically those models.

For each model tested, different horizons of time were adopted to calculate the VaR measure. The purpose is to verify whether different choices of length of time affect the results. The horizons of period assumed for estimation of VaR are 1, 2, 3, 4, and 5 years. The

VaR values are updated daily and the data used for estimation also moves accordingly. Then each actual daily return is compared to the daily VaR measure, giving a frequency of tail losses.

The previous section presented the mathematical definitions of VaR measures for parametric models. These models require a distribution assumption. If the empirical observations approximately coincide with the theoretical distribution assumed, then it is expected that the model perform well in forecasting risk. Otherwise, if the assumption is not appropriate, there is no reason to believe that the risk measures are accurate.

In this study, the first parametric VaR calculated assumed that returns are normally distributed. In order to be more coherent with the normal assumption, returns are taken in the logarithm form. As seen before, the normal VaR can be obtained by:

$$VaR^{R} = R_{cl}^{*} = \mu_{R} + \alpha_{cl}\sigma_{R} \tag{23}$$

The actual values of the parameters are not known, so VaR estimates are based on estimates of the parameters. Mean and standard deviation estimates are calculated in the usual way. The length of time used to estimate the standard deviation was 150 days. The mean, however, corresponds to length of time chosen to estimate VaR (1, 2, 3, 4, or 5 years). The standard normal variate, α_{cl}, corresponding to the chosen confidence level, is obtained directly from statistic tables. For 95% confidence level, the value of α_{cl} is -1.645, and for 99% *cl* is –2.326.

An alternative parametric approach selected was using the assumption that log returns follow a *t* distribution. According to Wilson (1993) this assumption fits better when working with log returns of financial assets. The estimate VaR formula, in this case, is very similar to the normal one except that it includes an additional multiplier term and $\alpha_{cl,v}$ now refers to a Student *t*-distribution and depends on *v* as well as *cl*. The *t*-VaR formula is:

$$VaR^R = R^*_{cl,v} = \mu_R + \alpha_{cl,v}\sqrt{(v-2)/v}\sigma_R \tag{24}$$

The number of degrees of freedom, v, assumed is 10 and hence the critical values for the 95% and 99% probability levels are -2.23σ and -3.17σ, respectively. The number of degrees of freedom follows the same as used by Penza and Bansal (2001, p.155) when studying parametric VaR estimates for market indexes (MIBTEL, DAX, NYSE). Mean and standard deviation are the same as calculated when considering normal distribution of returns.

The VaR estimation using the non-parametric historical simulation approach is much simpler. It consists basically in assembling the histogram of frequency and taking the observation corresponding to one minus the chosen confidence level. So, when we are interested in estimating the VaR measure for the 95% confidence level, we simply take the observation that corresponds to the 5% tail loss. For example, for the case of one-year period of estimation (247 observations), the VaR measure, at the 95% confidence level, corresponds to the 13th (247*0.05, approximately) smallest return in this sample. It is easy to note that in this approach all observations included in the estimation are given the same weight no matter its age. So long past observations have the same impact on estimation as near past observations. This fact brings the problem of ghost effects becoming the VaR estimation less responsive to market changes. Additionally, this approach does not consider that the market volatility varies over time.

In order to incorporate these aspects, we weight the data by volatility as suggested by Hull and White (1998a). This approach consists in updating return information by taking account recent changes in volatility. Returns used to estimate VaR become volatility-adjusted as follows:

$$r^*_{T,i} = \frac{\sigma_{T,i} \cdot r_{T,i}}{\sigma_{t,i}} \tag{25}$$

Therefore, actual returns in any period T are increased (or decreased) depending on whether the current forecast of volatility is greater (or less than) the estimated volatility for period t.[23] The gap in volatility of one month was used to adjust returns.

Because it is more plausible to assume that recent observations are more important in forecasting the near future risk than old observations, we can add in this aspect by means of the volatility estimation. For that, we estimate volatility using the exponentially weighted moving average (EWMA) method. This technique places more weight on recent observations, reducing the ghost effects, and also captures volatility clusters. The EWMA variance estimate was calculated as follows:

$$\sigma_t^2 = (1-\lambda)\sum_{i=1}^{n}\lambda^{i-1}r_{t-i}^2 \tag{26}$$

Attaining the volatility estimates through the formula above, we easily calculate the volatility-adjusted returns and, then, proceed as described for the historical simulation approach to obtain the VaR estimates.

3.3 Supporting tests

Before extracting conclusions from preliminary outputs, it is important to look for tests that give support and legitimacy to the results. The supporting tests applied in this study are hereafter described.

Because the parametric approach assumes some theoretical distribution to estimate VaR, it is a good idea first to conduct a test that evaluates the distribution of the observed returns. A widely used test is the Kolmogorov-Smirnov test. It compares an observed cumulative distribution function to a theoretical cumulative distribution. Parameters of the theoretical distribution are estimated from the observed data. In this study, the normal distribution is

[23] Dowd (2002, p.68)

selected. Although, the test may reject the distribution assumed the parametric VaR is still considered to confirm if the inadequacy of the assumption is reflected in the empirical results. Additionally, the kurtosis and the skew parameters are calculated in order to obtain a better characterization of the empirical distribution.

In order to verify the prediction correctness of each model, the binomial test (Kupiec (1995)) is applied. Using the failure rate, which is given by the proportion of times VaR is exceeded, this test evaluates whether the number of exceptions is acceptably small. Basically, the binomial test compares the observed values to the hypothesized values. So, to implement Kupiec's test we first need to modify the observations to a binomial framework. In this case, each observation exceeding VaR takes form of one and non-exceeding observations becomes zero. The sequence of failures x follows a binomial probability distribution given by:

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x} \tag{27}$$

Where n is the number of observations and p is the predicted frequency of tail losses. Therefore, the expected value of x is pn and its variance is given by $p(1-p)n$.

If n is large enough, by the central limit theorem, the binomial distribution can be approximated to the normal distribution. Then follows that:

$$z = \frac{x - pn}{\sqrt{p(1-p)n}} \approx N(0,1) \tag{28}$$

The null hypothesis tested is "the model gives good prediction about the number of losses that one should expect to exceed the VaR measure."

However, the binomial test only focuses in the frequency of the tail losses. The sizes of the losses are not evaluated by this test. Nevertheless, the VaR measure doesn't try to estimate the loss when it is beyond its value. This is out off its scope. On the other hand, the CVaR measure gives the expected loss given the loss is greater than VaR. Therefore, this measure is appropriate to be included in the test involving the size of the losses.

The test used to evaluate the effectiveness of the CVaR measure in predicting the value of the losses when these are greater than VaR was the t-test for paired samples differences. This test determines whether two samples are likely to have come from the same two underlying populations that have the same mean. The procedure used to apply this test was as follows. First, the mean value of the losses that exceeded the VaR measure was calculated. Then, each value of CVaR corresponding to the days in which the losses were greater than VaR was taken and its mean was obtained. Finally, the t-test for paired samples was applied in order to verify if the two means are statistically different.

4. Results

This section presents the empirical results attained in this study. First, it is presented an analysis about the data used. Following are the performance results of the models selected to measure risk in the context of daily returns of the Brazilian stock market index. It is shown the accuracy of parametric and non-parametric approaches in forecasting the maximum loss expected given a confidence level (VaR) and the ability of these models in predicting the value of losses exceeding VaR (CVaR).

4.1 Data analysis

Lets take a look on the distribution of the daily returns during the period selected (Jan/1995 through Dec/2004). This information might give a good indication of what we might expect from the parametric models selected. Figure 5 shows the histogram of daily log returns for the Brazilian market index during the 10-year period of test. A normal distribution has been fit to the data based upon the data's sample mean and sample standard deviation.

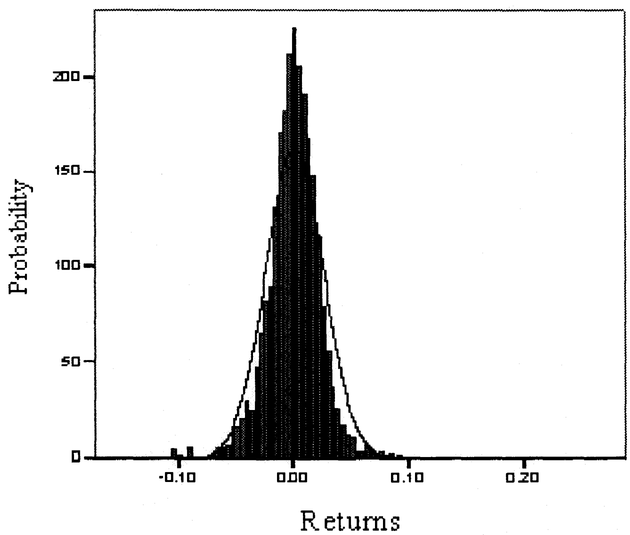

Figure 5: Histogram of daily log returns for the Brazilian market index during the 10-year period 1995 through 2004. A normal distribution has been fit to the data based upon its sample mean and sample standard deviation.

Comparing the normal curve and the histogram, it can be noted that the histogram is leptokurtic. The Kolmogorov-Smirnov test also confirms that the distribution is not normal. The empirical distribution has a higher concentration of returns in the central area than the normal distribution predicts. This might lead us to believe that it has a lower standard deviation. However, it has also fatter tails, which goes in the opposite direction. Table 1 compares the mean, standard deviation, skewness, and kurtosis of the normal curve and histogram.

Table 1: Parameters of the normal distribution compared with sample parameters of the histogram of log returns for the Brazilian market index.

	Normal Distribution	Histogram
Mean	0.072%	0.072%
Standard Deviation	2.533%	2.533%
Skewness	0.00	0.61
Kurtosis	3.00	13.29

The kurtosis found for the histogram was 13.29, confirming that the distribution is more peaked at the center and has fatter tails. The measure of asymmetry of the distribution (skewness) found was 0.61, which suggests that the empirical distribution is slightly asymmetric. The positive number indicates the presence of a short tail in the left and a long tail on the right. These aspects suggest that the *t*-distribution might be more adequate to model the VaR measure. However, it is important to note that this histogram represents the distribution of returns for the entire period and the VaR estimates are obtained from histograms assembled with partial data. So, not necessarily, the partial data histograms will have the same characteristics of the histogram presented above and the normal assumption might be still a better approach to use.

Observing the sample distribution statistics for different horizon periods (table 2), it can be noted that there is extreme variation of the kurtosis and skewness parameters for all periods of estimation. For example, histograms assembled with 1-year sample period had a minimum kurtosis of 0.449 and a maximum of 44.262, and a minimum skewness of –1.520

and a maximum of 4.518. The highest standard deviation for the kurtosis parameter was observed for 3-year histograms, and for 2-year histograms for the skewness parameter.

Table 2: Parameters of the distributions assembled with log returns of the Brazilian market index for different horizon periods (1, 2, 3, 4, and 5 years).

Length	Kurtosis				Skewness			
of period	Mean	St. Dev.	Minimum	Maximum	Mean	St. Dev.	Minimum	Maximum
1 year	3.304	5.660	0.449	44.262	0.003	0.768	-1.520	4.518
2 years	5.744	6.799	0.249	37.031	0.091	0.805	-1.332	3.377
3 years	7.897	6.924	0.409	30.184	0.234	0.701	-1.285	2.491
4 years	10.206	6.788	0.749	25.440	0.469	0.527	-0.338	2.011
5 years	12.116	5.923	0.787	25.697	0.560	0.444	-0.231	1.871

These facts illustrate that the distribution characteristics vary over the period of analysis. Same horizon period distributions give signs of fat and thin tails depending on the moment the parameter is taken. The asymmetric parameter also changes over time. These aspects of the distributions indicate that parametric approaches might be not suitable to model risk for the Brazilian market index.

4.2 Normal VaR (N-VaR)

As stated in the previous section, the binomial test (Kupiec (1995)) can test the effectiveness of each model in predicting the number of losses exceeding the VaR measure. Table 3 presents the results for the parametric approach assuming returns are normally distributed, at the 95% and 99% confidence levels. Once again, different horizons for estimation were used.

The results show that, for the chosen 95% confidence level, the normal parametric approach was adequate (sig.>0.05) to measure risk when 1, 2, and 3-year periods of estimation were used. In these cases, losses exceeded predicted VaR between 5-6% of occasions. On the other hand, the binomial test rejected the model when 4 and 5-year periods of estimation were applied. Losses exceeded predicted normal VaR on only 3% of occasions in both cases, suggesting that the model overstated 'true' VaRs. This result is in accordance with the empirical distribution shown before, suggesting that the partial data distributions differ from the entire data distribution.

Table 3: Results of binomial test for the Normal VaR at the 95% and 99% confidence levels and different horizons of estimation (1, 2, 3, 4, and 5 years).

Period of estimation Panel A: 95% *cl*		Category	N	Observed Prop.	Test Prop.	Asymp. Sig. (1-tailed)
1 year N-VaR	Group 1	≤ VaR	2103	0.943	0.95	0.061[a,c]
	Group 2	> VaR	128	0.057		
	Total		2231	1.000		
2 year N-VaR	Group 1	≤ VaR	1868	0.942	0.95	0.057[a,c]
	Group 2	> VaR	115	0.058		
	Total		1983	1.000		
3 year N-VaR	Group 1	≤ VaR	1650	0.952	0.95	0.404[c]
	Group 2	> VaR	84	0.048		
	Total		1734	1.000		
4 year N-VaR	Group 1	≤ VaR	1441	0.968	0.95	0.001[c]
	Group 2	> VaR	47	0.032		
	Total		1488	1.000		
5 year N-VaR	Group 1	≤ VaR	1206	0.971	0.95	0.000[c]
	Group 2	> VaR	36	0.029		
	Total		1242	1.000		
Panel B: 99% *cl*						
1 year N-VaR	Group 1	≤ VaR	2186	0.980	0.99	0.000[b,c]
	Group 2	> VaR	45	0.020		
	Total		2231	1.000		
2 year N-VaR	Group 1	≤ VaR	1938	0.977	0.99	0.000[b,c]
	Group 2	> VaR	45	0.023		
	Total		1983	1.000		
3 year N-VaR	Group 1	≤ VaR	1703	0.982	0.99	0.001[b,c]
	Group 2	> VaR	31	0.018		
	Total		1734	1.000		
4 year N-VaR	Group 1	≤ VaR	1478	0.993	0.99	0.127[c]
	Group 2	> VaR	10	0.007		
	Total		1488	1.000		
5 year N-VaR	Group 1	≤ VaR	1233	0.993	0.99	0.202[c]
	Group 2	> VaR	9	0.007		
	Total		1242	1.000		

a Alternative hypothesis states that the proportion of cases in the first group < .95.
b Alternative hypothesis states that the proportion of cases in the first group < .99.
c Based on Z Approximation.

Interestingly, the results observed for normal parametric VaR, at the chosen higher confidence level of 99% (panel B), were reversed. The model was rejected when short periods of estimation were used (1, 2, and 3 years) and was accepted when long periods of estimation were adopted (4 and 5 years). Short period of estimation provided low VaRs, resulting in the number of losses exceeding this parameter being greater than predicted.

To understand the reverse on the results for normal parametric VaR when the chosen confidence level increased, take a look on the graphs of the time series of both daily log returns and VaRs. Following are the graphs representing the series observed for 1 and 5 year period of estimation (figures 6 and 7, respectively).

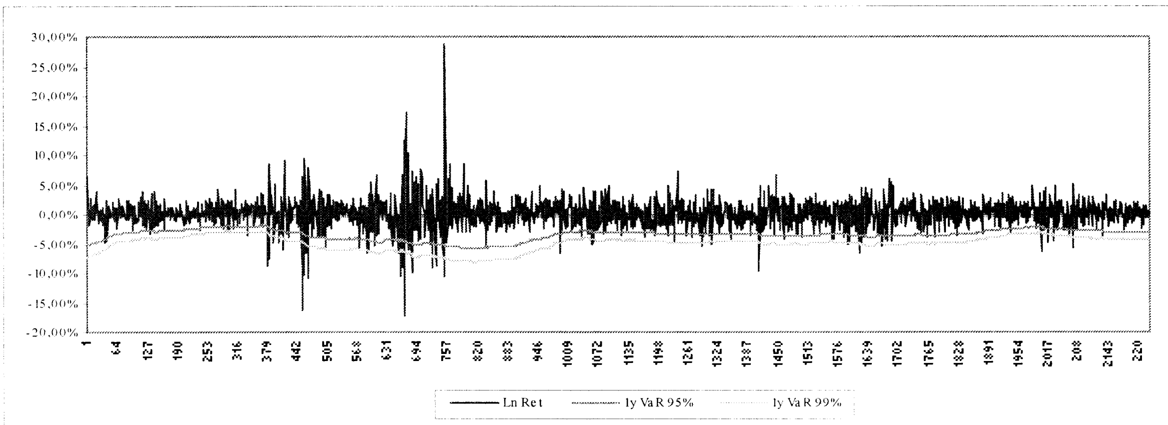

Figure 6: Time series of both daily log returns and N-VaRs for 1-year period of estimation.

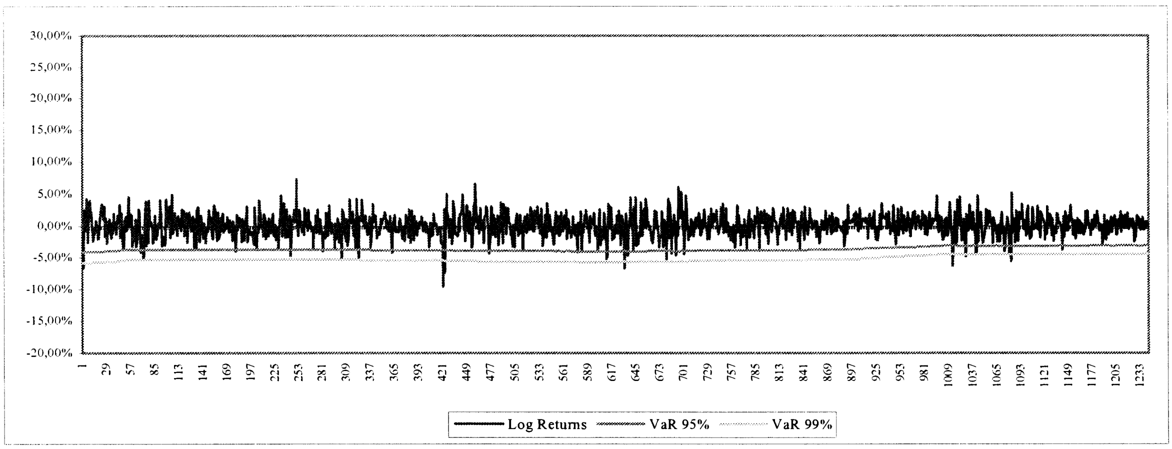

Figure 7: Time series of both daily log returns and N-VaRs for 5-year period of estimation.

As can be observed, the VaR estimate using 5 years for estimation is smoother than when using 1 year for estimation. The 5-year estimates are almost a straight line, not responding to changes in standard deviation. Its estimates have the high standard deviation data in its VaR for a long period. This might explain the overestimation of VaR, at the 95% confidence level, which happened when large periods of estimation were used. Contrary, the 1-year estimate is more responsive to changes in standard deviation. This fact can clearly be seen between observations 700 and 1,000 (figure 6), when the VaR estimate increased considerably, incorporating the high standard deviation data, and afterwards decreased when this data gets aged.

However, how to explain the results when the 99% confidence level was chosen? The answer is the period in which the model was tested. High volatile periods make more difficult to forecast risk measures at higher *cl*. This fact was confirmed when different horizons of estimation were tested (at high *cl*) but now only in the second half of the sample period (observation 1115 to 2231). At this time, all estimations passed the binomial test. The same did not happen at the 95% *cl*, when the results remained the same as found before.

Therefore, the results for the normal VaR show that at higher confidence level the model is highly dependent to the period of test, being more accurate in low volatile periods no matter the length of time used for estimation. At the 95% *cl*, however, the length of time used for estimation matters more than the period of test. Short periods of estimation give more accurate VaR estimates in this case.

4.3 Student t VaR (T-VaR)

The results of the binomial test for the T-VaR estimates, at the 95 and 99% *cl*, are presented in table 4.

Clearly, using a *t*-distribution overestimated the VaR measures. For all rejected cases, the number of losses exceeding T-VaRs was lower than predicted. Nevertheless, for 1, 2, and

3-year periods of estimation (at the 99% *cl*), the model corrected the underestimation provided by the normal VaR and the T-VaR demonstrated to be an adequate measure of risk in these cases. Overall, the results for the T-VaR are not in accordance with Wilson (1993) who proposed that financial assets log returns should be modeled with student *t*-distributions.

Table 4: Results of binomial test for the T-VaR at the 95% and 99% confidence levels and different horizons of estimation (1, 2, 3, 4, and 5 years).

Period of estimation Panel A: 95% *cl*		Category	N	Observed Prop.	Test Prop.	Asymp. Sig. (1-tailed)
1 year T-VaR	Group 1	≤ VaR	2166	0.971	0.95	0.000 [a]
	Group 2	> VaR	65	0.029		
	Total		2231	1.000		
2 year T-VaR	Group 1	≤ VaR	1920	0.968	0.95	0.000 [a]
	Group 2	> VaR	63	0.032		
	Total		1983	1.000		
3 year T-VaR	Group 1	≤ VaR	1693	0.976	0.95	0.000 [a]
	Group 2	> VaR	41	0.024		
	Total		1734	1.000		
4 year T-VaR	Group 1	≤ VaR	1473	0.990	0.95	0.000 [a]
	Group 2	> VaR	15	0.010		
	Total		1488	1.000		
5 year T-VaR	Group 1	≤ VaR	1229	0.990	0.95	0.000 [a]
	Group 2	> VaR	13	0.010		
	Total		1242	1.000		
Panel B: 99% *cl*						
1 year T-VaR	Group 1	≤ VaR	2210	0.991	0.99	0.432 [c]
	Group 2	> VaR	21	0.009		
	Total		2231	1		
2 year T-VaR	Group 1	≤ VaR	1958	0.987	0.99	0.146 [b,c]
	Group 2	> VaR	25	0.013		
	Total		1983	1		
3 year T-VaR	Group 1	≤ VaR	1721	0.993	0.99	0.177 [c]
	Group 2	> VaR	13	0.007		
	Total		1734	1		
4 year T-VaR	Group 1	≤ VaR	1484	0.997	0.99	0.003 [c]
	Group 2	> VaR	4	0.003		
	Total		1488	1		
5 year T-VaR	Group 1	≤ VaR	1239	0.998	0.99	0.005 [c]
	Group 2	> VaR	3	0.002		
	Total		1242	1		

a Alternative hypothesis states that the proportion of cases in the first group < .95.
b Alternative hypothesis states that the proportion of cases in the first group < .99.
c Based on Z Approximation.

4.4 Historical Simulation VaR (HS-VaR)

The first non-parametric model tested to estimate risk was the historical simulation approach. Table 5 shows the results of the binomial test for the HS-VaR.

Table 5: Results of binomial test for the HS-VaR at the 95% and 99% confidence levels and different horizons of estimation (1, 2, 3, 4, and 5 years).

Period of estimation Panel A: 95% *cl*		Category	N	Observed Prop.	Test Prop.	Asymp. Sig. (1-tailed)
1 year HS-VaR	Group 1	≤ VaR	2114	0.948	0.95	0.315 [a,c]
	Group 2	> VaR	117	0.052		
	Total		2231	1.000		
2 year HS-VaR	Group 1	≤ VaR	1884	0.950	0.95	0.514 [c]
	Group 2	> VaR	99	0.050		
	Total		1983	1.000		
3 year HS-VaR	Group 1	≤ VaR	1658	0.956	0.95	0.131 [c]
	Group 2	> VaR	76	0.044		
	Total		1734	1.000		
4 year HS-VaR	Group 1	≤ VaR	1436	0.965	0.95	0.005 [c]
	Group 2	> VaR	52	0.035		
	Total		1488	1.000		
5 year HS-VaR	Group 1	≤ VaR	1212	0.976	0.95	0.000 [c]
	Group 2	> VaR	30	0.024		
	Total		1242	1.000		
Panel B: 99% *cl*						
1 year HS-VaR	Group 1	≤ VaR	2201	0.987	0.99	0.063 [b,c]
	Group 2	> VaR	30	0.013		
	Total		2231	1.000		
2 year HS-VaR	Group 1	≤ VaR	1958	0.987	0.99	0.146 [b,c]
	Group 2	> VaR	25	0.013		
	Total		1983	1.000		
3 year HS-VaR	Group 1	≤ VaR	1718	0.991	0.99	0.420 [c]
	Group 2	> VaR	16	0.009		
	Total		1734	1.000		
4 year HS-VaR	Group 1	≤ VaR	1479	0.994	0.99	0.080 [c]
	Group 2	> VaR	9	0.006		
	Total		1488	1.000		
5 year HS-VaR	Group 1	≤ VaR	1239	0.998	0.99	0.005 [c]
	Group 2	> VaR	3	0.002		
	Total		1242	1.000		

a Alternative hypothesis states that the proportion of cases in the first group < .95.
b Alternative hypothesis states that the proportion of cases in the first group < .99.
c Based on Z Approximation.

At the 95% *cl*, the results are similar to found for the normal VaR. Using horizon period of 1, 2, and 3 years for estimation the model presented to be adequate to predict risk of the daily Brazilian stock index returns. For 4 and 5-year periods of estimation the model was rejected. At the 99% *cl*, the results are the same except that at this time the 4-year period of estimation was also approved.

Now, check the graphs of the time series of both daily returns and HS-VaRs for 1 and 5-year periods of estimation to find the reasons behind these results.

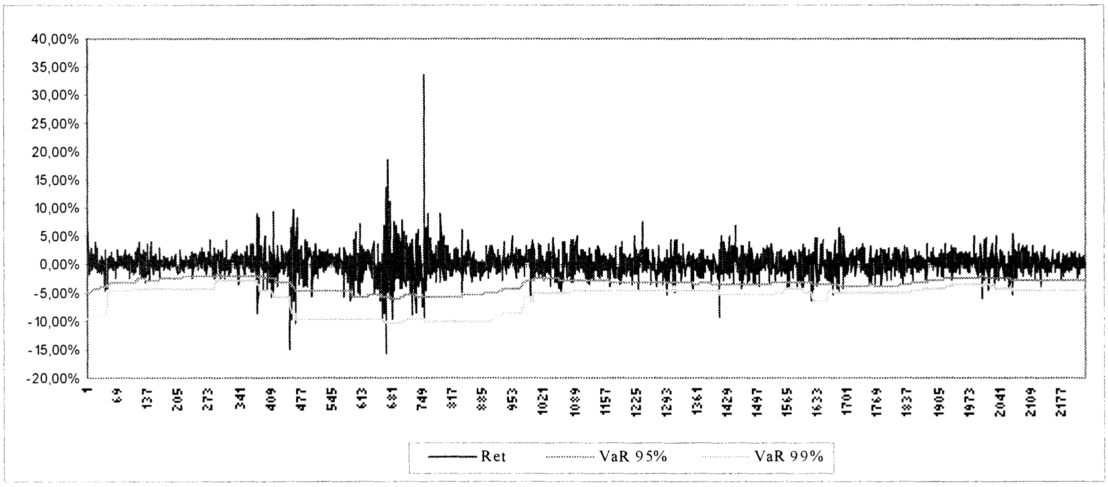

Figure 8: Time series of both daily returns and HS-VaRs for 1-year period of estimation.

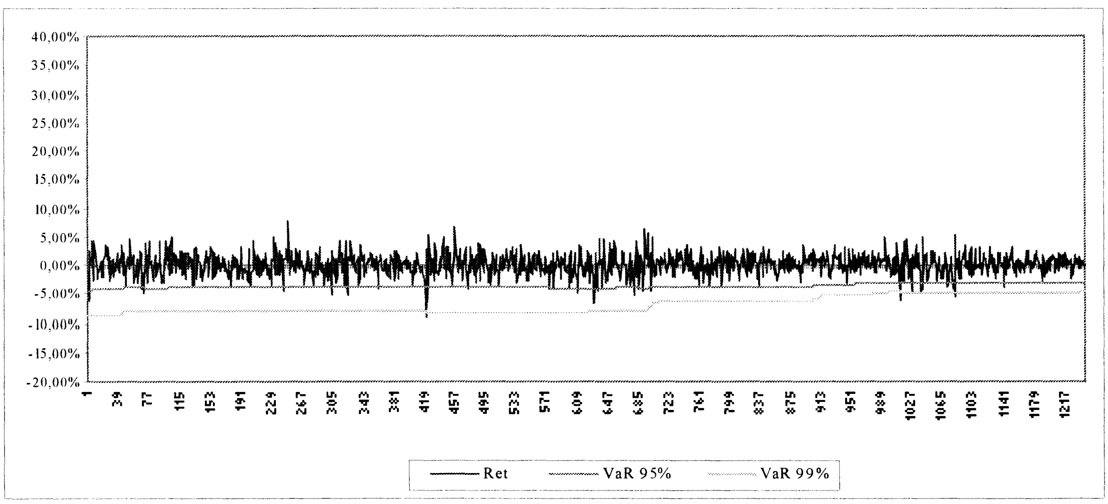

Figure 9: Time series of both daily returns and HS-VaRs for 5-year period of estimation.

The behavior of the HS-VaR estimates is similar to found for the N-VaR estimates. The difference is that now the VaRs are taken in discrete form and so there are some jumps in the values. As for the N-VaR, as we move from 1 to 5 years period of estimation the HS-VaR estimate becomes less responsive to market volatility changes. For 5-year period of estimation, the risk lines show flatness, or excessive smoothness. This means that the risk measures are not being updated sufficiently quickly. At the 95% *cl*, the VaR estimates are too high for the risks one would be actually facing. The rate of losses exceeding VaR was only 3%. Again, at the 99% *cl*, the VaR estimates are too high and the rate of losses exceeding VaR was very close to 0%.

These last results are direct consequences of ghost effects. Unlikely to recur losses are incorporated in the sample and dominate the HS risk estimates until the data gets aged. This is an important drawback of the simple historical simulation approach. Past returns entering into the VaR estimative are equally important and long ago returns have the same weight as recent observations. Further, high confidence levels means small tails, more distant cut-off points defining VaR, which are more affected by extreme losses.

4.5 Volatility-Weighted Historical Simulation VaR (VWHS-VaR)

Another problem with the simple historical simulation approach is that it assumes that volatility is constant over time. This is obviously not the case when dealing with daily stock returns. Figure 10 shows the changes in volatility (standard deviation) over time for the time period used in this study.

Clearly, volatility varies over time. The first half of the period is characterized by higher volatilities than the second half. The first half volatility statistics is: minimum volatility of 0.95%, maximum volatility of 6.34%, mean of 2.79%, and standard deviation of 1.40%. For the second half we have: minimum volatility of 1.15%, maximum volatility of 2.60%, mean of 1.92%, and standard deviation of 0.32%. Comparing these numbers undoubtedly confirm the visual evidence shown by the graph above.

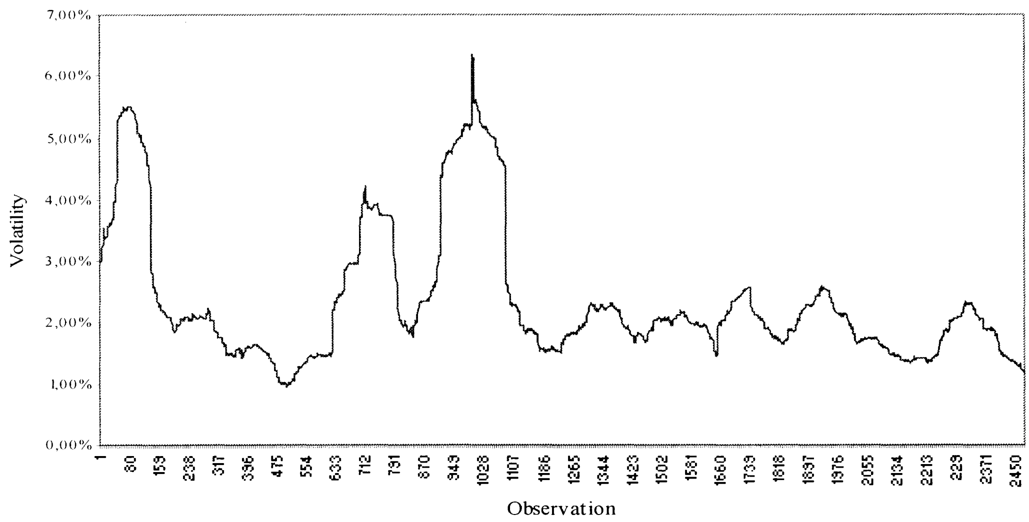

Figure 10: Changes in volatility (standard deviation) for the Brazilian Stock Market Index over the period Jan/1994 to Dec/2004.

To circumvent most of the problems of the VaR estimates when normal and *t* parametric approaches and the simple historical simulation approach were adopted, we used a more sophisticated tool to measure VaR. This consists in giving weights to the data by volatility, which is calculated using a moving average scheme with declining weights. The results of the binomial test applied to the VaR estimates given by this model (VWHS-VaR), at the 95% and 99% confidence levels, are presented in table 6.

Undoubtedly, the VWHS-VaR model had much better performance than the previous models. For all horizon periods for estimation and confidence levels the binomial test approved the VaR predictions made by the model. The rates of losses exceeding VaR measures ranged between 4.7-6.0%, at the 95% *cl*, and between 0.9-1.2%, at the 99% *cl*. Further, when the test was conducted only for the second half of the test period (lower volatility period) the results remained unchanged. The length of period for estimation and the confidence level chosen don't have large influence on the results attained by this model. This fact indicates more confidence on the risk estimates given by this approach.

Table 6: Results of binomial test for the VWHS-VaR at the 95% and 99% confidence levels and different horizons of estimation (1, 2, 3, 4, and 5 years).

Period of estimation Panel A: 95% *cl*		Category	N	Observed Prop.	Test Prop.	Asymp. Sig. (1-tailed)
1 year VWHS-VaR	Group 1	≤ VaR	2127	0.953	0.95	0.247 [a,c]
	Group 2	> VaR	104	0.047		
	Total		2231	1.000		
2 year VWHS-VaR	Group 1	≤ VaR	1875	0.946	0.95	0.195 [a,c]
	Group 2	> VaR	108	0.054		
	Total		1983	1.000		
3 year VWHS-VaR	Group 1	≤ VaR	1646	0.949	0.95	0.465 [a,c]
	Group 2	> VaR	88	0.051		
	Total		1734	1.000		
4 year VWHS-VaR	Group 1	≤ VaR	1410	0.948	0.95	0.356 [a,c]
	Group 2	> VaR	78	0.052		
	Total		1488	1.000		
5 year VWHS-VaR	Group 1	≤ VaR	1168	0.940	0.95	0.069 [a,c]
	Group 2	> VaR	74	0.060		
	Total		1242	1.000		
99% *cl*						
1 year VWHS-VaR	Group 1	≤ VaR	2210	0.991	0.99	0.432 [b,c]
	Group 2	> VaR	21	0.009		
	Total		2231	1.000		
2 year VWHS-VaR	Group 1	≤ VaR	1960	0.988	0.99	0.273 [b,c]
	Group 2	> VaR	23	0.012		
	Total		1983	1.000		
3 year VWHS-VaR	Group 1	≤ VaR	1716	0.990	0.99	0.485 [b,c]
	Group 2	> VaR	18	0.010		
	Total		1734	1.000		
4 year VWHS-VaR	Group 1	≤ VaR	1474	0.991	0.99	0.461 [b,c]
	Group 2	> VaR	14	0.009		
	Total		1488	1.000		
5 year VWHS-VaR	Group 1	≤ VaR	1228	0.989	0.99	0.379 [b,c]
	Group 2	> VaR	14	0.011		
	Total		1242	1.000		

a Alternative hypothesis states that the proportion of cases in the first group < .95.
b Alternative hypothesis states that the proportion of cases in the first group < .99.
c Based on Z Approximation.

However, observing the time series of both daily returns and VWHS-VaRs for 1 and 5 year period of estimation (figures 11 and 12, respectively), we can note that the model is not completely absent of the ghosts effects. Large losses affects the VaR estimates and they remain until the volatility-weighted measure corrects VaR, which depends on the magnitude of the loss and the change caused in volatility. The model predicts that subsequent losses can be larger when there is an increase in volatility. This characteristic of the model leads sometimes to overstate future losses. This feature is more prominent at the 99% confidence level, which it is not surprising because this risk measure relies on the very extreme losses.

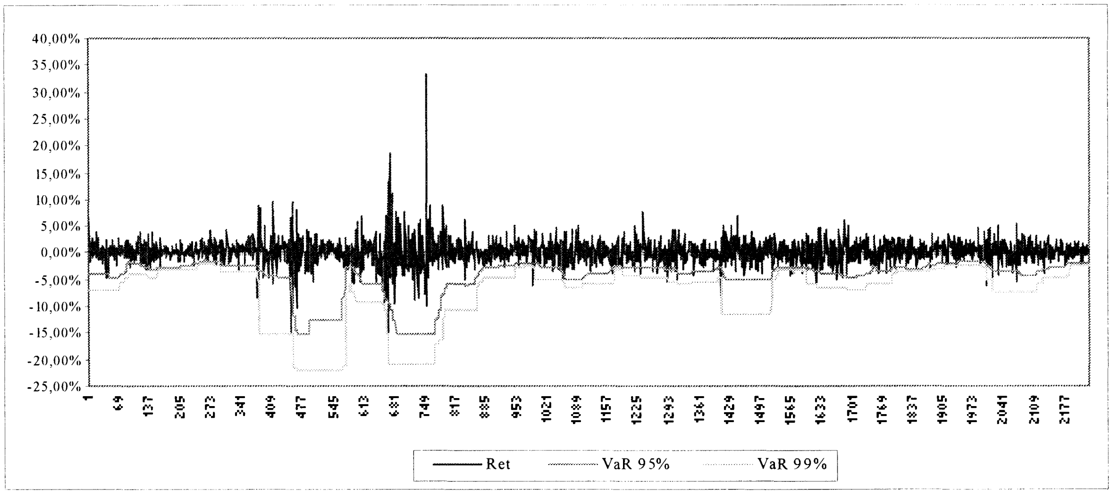

Figure 11: The time series of both daily returns and VWHS-VaRs for 1-year period of estimation.

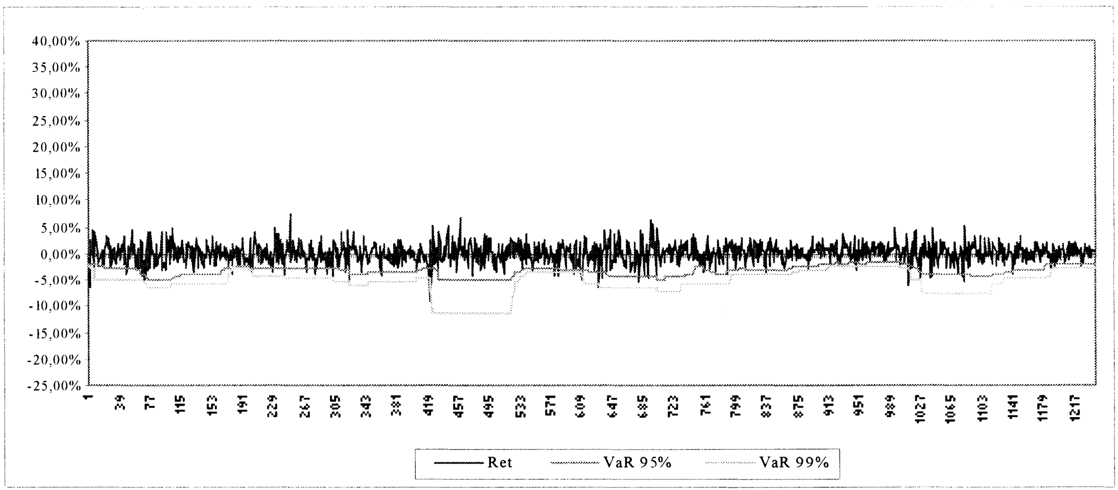

Figure 12: The time series of both daily returns and VWHS-VaRs for 5-year period of estimation.

So far, the results indicate that considering the relative importance of old observations and changes in volatility give superior VaR estimates of daily returns of the Brazilian market index. In this specific case, the ability to accommodate fat tails and skewness of the return distributions was the primary source of success in estimating the risk one would be facing and this was better reached by the nonparametric approach of volatility-weighted historical simulation. Lets see if the results for the CVaR estimates remain the same. Next section presents the results.

4.6 Normal CVaR (N-CVaR)

This section presents the results for the test of accuracy of the CVaR measures, made by the parametric approach assuming returns are normally distributed, in forecasting the value of losses beyond VaR. The paired samples *t* test was applied using the values of the losses that exceeded the N-VaR measure and the corresponding estimated N-CVaR measure.

Before start analyzing the results, a note is herewith necessary. Sometimes, especially when working with a high confidence level like 99%, the number of cases in which the test is applied is very small and so the results attained are not reliable. In this section, we will focus only on the results in which the number of cases applied is large enough to provide reliable results, for instance N > 30.

Table 7 shows the results for the paired samples *t* test applied between the losses that exceeded the N-VaR measure and the corresponding N-CVaR estimates. It points out that there is no significant difference between the N-CVaR measures and the losses exceeding the N-VaR for 4 and 5 years period of estimation at the 95% confidence level. In these cases, paired differences ranged from a lower of –0.73% and an upper of 0.12%, with a mean of –0.30%. The paired correlation result, however, was significant only for 4-year period of estimation. The correlation found was 0.459. Since the two variables represent two related groups, the correlation should be fairly high which in this case is not necessarily true.

Importantly, it can be noted that, for the cases in which the means were statistically different, the paired samples differences mean was negative in all cases. This indicates that the 'true' value of the losses that exceeded the VaR measures were, on average, smaller than predicted by the N-CVaR. The model overestimated these measures.

Table 7: Paired samples statistics, correlations, differences, and *t* test for the pairs of losses exceeding N-VaR measures and estimated N-CVaR measures at the 95 and 99% confidence levels and different horizon periods of estimation (1, 2, 3, 4, and 5 years).

95% *cl*		Paired Samples Statistics			Paired Correlations		Paired Differences				Paired Test	
Pair	Period of est.	Mean	N	Std. Dev.	Correlation	Sig.	Mean	Std. Dev.	Lower (95% *cl*)	Upper(95% cl)	t	Sig. (2-tailed)
Loss > N-VaR	1 year	-4.86%	120	2.32%	0.530	0.000	-0.77%	1.98%	-1.12%	-0.41%	-4.238	0.000
N-CVaR	1 year	-4.09%	120	1.02%								
Loss > N-VaR	2 years	-5.25%	105	2.26%	0.240	0.013	-1.03%	2.20%	-1.45%	-0.60%	-4.795	0.000
N-CVaR	2 years	-4.22%	105	0.66%								
Loss > N-VaR	3 years	-5.24%	80	1.97%	0.178	0.113	-0.78%	1.94%	-1.21%	-0.35%	-3.598	0.001
N-CVaR	3 years	-4.46%	80	0.42%								
Loss > N-VaR	4 years	-5.01%	42	1.40%	0.459	0.002	-0.33%	1.25%	-0.72%	0.06%	-1.701	0.096
N-CVaR	4 years	-4.68%	42	0.48%								
Loss > N-VaR	5 years	-4.92%	31	1.15%	0.167	0.370	-0.30%	1.16%	-0.73%	0.12%	-1.463	0.154
N-CVaR	5 years	-4.62%	31	0.42%								
99% *cl*												
Loss > N-VaR	1 year	-6.68%	45	2.75%	0.590	0.000	-1.39%	2.23%	-2.05%	-0.72%	-4.171	0.000
N-CVaR	1 year	-5.29%	45	1.42%								
Loss > N-VaR	2 years	-7.07%	43	2.52%	0.294	0.055	-1.68%	2.42%	-2.43%	-0.94%	-4.565	0.000
N-CVaR	2 years	-5.39%	43	0.88%								
Loss > N-VaR	3 years	-7.04%	28	2.39%	0.400	0.035	-1.44%	2.24%	-2.30%	-0.57%	-3.398	0.002
N-CVaR	3 years	-5.61%	28	0.50%								
Loss > N-VaR	4 years	-7.03%	9	1.68%	0.812	0.008	-1.05%	1.18%	-1.95%	-0.14%	-2.654	0.029
N-CVaR	4 years	-5.99%	9	0.71%								
Loss > N-VaR	5 years	-6.53%	7	1.41%	0.584	0.168	-0.79%	1.16%	-1.86%	0.28%	-1.796	0.123
N-CVaR	5 years	-5.75%	7	0.67%								

Comparing the N-CVaR and the N-VaR results, we note that the VaR is better estimated when short periods of estimation were used while for the CVaR estimates better results were attained when long periods of estimation were used. However, the CVaR estimate is directly associated with the VaR estimate, which turns the results for the CvaR ambiguous. Although, the result of the paired samples test for N-CVaR (4 years of estimation) was significant, the N-CVaR measures used were obtained from N-VaR measures that were established, through the binomial test, not being good measures of risk. So, this fact can disqualify the result of the N-CVaR in this case.

4.7 Student-t CVaR (T-CVaR)

The results for the paired samples *t* test applied between the losses that exceeded the T-VaR measure and the corresponding T-CVaR estimates are presented in table 8. Considering the minimum of 30 observations, the only chosen period of estimation and confidence level that there was no significant difference between the T-CVaR measures and the losses exceeding the T-VaR was 3-year period of estimation at the 99% confidence level. The paired differences ranged from –0.46% to 1.25%, with a mean of 0.40%. The paired correlation, in this case, was also significant. The correlation found was 0.360, which can be considered low under the circumstances.

Additionally, for most cases in which the paired differences were significant, the mean of the differences were negative and greater than found for normal CVaR. This means that the *t*-distribution assumption caused even greater overestimation of the values of the losses beyond its VaR.

When comparing the T-CVaR and the T-VaR results, at this time, there is no ambiguity between the results. The T-VaR, using 3 years period of estimation at 99% confidence level, was also accepted by the binomial test. So the direct relation between the CVaR and VaR measure does not compromise the results in this case.

Table 8: Paired samples statistics, correlations, differences, and *t* test for the pairs of losses exceeding T-VaR measures and estimated T-CVaR measures at the 95 and 99% confidence levels and different horizon periods of estimation (1, 2, 3, 4, and 5 years).

95% *cl*		Paired Samples Statistics			Paired Correlations		Paired Differences				Paired Test	
Pair	Period of est.	Mean	N	Std. Dev.	Correlation	Sig.	Mean	Std. Dev.	Lower (95% *cl*)	Upper(95% cl)	t	Sig. (2-tailed)
Loss > T-VaR	1 year	-6.33%	48	3.20%	0.624	0.000	-0.96%	2.50%	-1.68%	-0.23%	-2.649	0.011
T-CVaR	1 year	-5.37%	48	1.88%								
Loss > T-VaR	2 years	-6.68%	53	2.89%	0.395	0.003	-1.29%	2.65%	-2.03%	-0.56%	-3.552	0.001
T-CVaR	2 years	-5.39%	53	1.24%								
Loss > T-VaR	3 years	-5.30%	96	2.00%	0.175	0.088	0.58%	2.08%	0.16%	1.00%	2.737	0.007
T-CVaR	3 years	-5.88%	96	1.00%								
Loss > T-VaR	4 years	-6.02%	18	2.22%	0.747	0.000	-0.40%	1.53%	-1.16%	0.37%	-1.096	0.288
T-CVaR	4 years	-5.63%	18	1.25%								
Loss > T-VaR	5 years	-5.25%	25	1.36%	0.666	0.000	0.03%	1.01%	-0.39%	0.45%	0.144	0.887
T-CVaR	5 years	-5.28%	25	0.90%								
99% *cl*												
Loss > T-VaR	1 year	-8.30%	17	3.83%	0.635	0.006	-1.54%	3.00%	-3.08%	0.00%	-2.114	0.051
T-CVaR	1 year	-6.77%	17	1.96%								
Loss > T-VaR	2 years	-8.52%	24	3.20%	0.405	0.050	-1.52%	2.95%	-2.76%	-0.27%	-2.524	0.019
T-CVaR	2 years	-7.00%	24	1.62%								
Loss > T-VaR	3 years	-7.19%	32	2.51%	0.360	0.043	0.40%	2.37%	-0.46%	1.25%	0.945	0.352
T-CVaR	3 years	-7.59%	32	1.30%								
Loss > T-VaR	4 years	-8.84%	4	2.19%	0.650	0.350	-0.88%	1.74%	-3.64%	1.88%	-1.012	0.386
T-CVaR	4 years	-7.96%	4	1.91%								
Loss > T-VaR	5 years	-6.13%	7	1.68%	0.936	0.002	0.37%	0.72%	-0.30%	1.04%	1.355	0.224
T-CVaR	5 years	-6.50%	7	1.16%								

4.8 Historical Simulation CVaR (HS-CVaR)

Now look on the results for the CVaR measure attained when the historical simulation approach was adopted (table 9). Although, for most of the chosen period of estimation and confidence level the difference between the expected tail loss and the actual tail loss was not considered statistically significant, when considering the minimum of 30 observations to validate the results, only the HS-CVaRs at the 95% *cl* using 1, 2, and 3-year periods of estimation are not rejected. Mean paired differences are very close among these cases, ranging from –0.26% to –0.28%. Additionally, the standard deviation found for these differences are lower than the ones found in previous models. This indicates that besides the differences tend to be small they do not vary too much.

Correlation between HS-CVaR and losses beyond HS-VaR, however, decreases as the period of estimation gets larger. For the two cases in which the paired test accused significant difference between the HS-CVaR measure and tail losses (4 and 5-year periods of estimation at the 95% *cl*), the paired mean differences indicate that the model underestimated the losses beyond its VaR.

Finally, jointly considering the results of the binomial test for the HS-VaR measure and the paired mean test for the HS-CVaR measure, we can consider that the historical simulation approach was an adequate model to estimate VaR and CVaR using short horizons (1, 2, and 3 years) for estimation. This approach provided good estimates of the maximum losses one should expect at a certain confidence level and the value of the tail losses as well.

Table 9: Paired samples statistics, correlations, differences, and *t* test for the pairs of losses exceeding HS-VaR measures and estimated HS-CVaR measures at the 95 and 99% confidence levels and different horizon periods of estimation (1, 2, 3, 4, and 5 years).

95% *cl*		Paired Samples Statistics			Paired Correlations		Paired Differences				Paired Test	
Pair	Period of est.	Mean	N	Std. Dev.	Correlation	Sig.	Mean	Std. Dev.	Lower (95% *cl*)	Upper(95% cl)	t	Sig. (2-tailed)
Loss > HS-VaR	1 year	-4.92%	107	2.42%	0.658	0.000	-0.26%	1.82%	-0.61%	0.09%	-1.473	0.144
HS-CVaR	1 year	-4.66%	107	1.63%								
Loss > HS-VaR	2 years	-5.32%	97	2.33%	0.405	0.000	-0.25%	2.16%	-0.69%	0.18%	-1.154	0.251
HS-CVaR	2 years	-5.06%	97	1.26%								
Loss > HS-VaR	3 years	-4.39%	127	1.93%	0.182	0.041	-0.28%	1.94%	-0.62%	0.06%	-1.637	0.104
HS-CVaR	3 years	-4.11%	127	0.79%								
Loss > HS-VaR	4 years	-5.04%	40	1.42%	0.520	0.001	0.51%	1.26%	0.11%	0.91%	2.552	0.015
HS-CVaR	4 years	-5.55%	40	1.06%								
Loss > HS-VaR	5 years	-4.96%	30	1.15%	0.194	0.304	0.79%	1.33%	0.29%	1.28%	3.245	0.003
HS-CVaR	5 years	-5.74%	30	0.92%								
99% *cl*												
Loss > HS-VaR	1 year	-7.42%	20	3.50%	0.732	0.000	-0.42%	2.60%	-1.64%	0.80%	-0.723	0.479
HS-CVaR	1 year	-7.00%	20	3.60%								
Loss > HS-VaR	2 years	-7.98%	20	3.19%	0.611	0.004	-0.59%	2.54%	-1.78%	0.60%	-1.033	0.314
HS-CVaR	2 years	-7.39%	20	2.29%								
Loss > HS-VaR	3 years	-7.03%	27	2.46%	0.442	0.021	-0.71%	2.27%	-1.61%	0.19%	-1.630	0.115
HS-CVaR	3 years	-6.32%	27	1.64%								
Loss > HS-VaR	4 years	-7.69%	4	2.22%	0.980	0.020	0.76%	0.78%	-0.49%	2.01%	1.936	0.148
HS-CVaR	4 years	-8.45%	4	2.82%								
Loss > HS-VaR	5 years	-6.93%	3	1.98%	0.981	0.781	0.76%	0.70%	-0.97%	2.49%	1.893	0.199
HS-CVaR	5 years	-7.69%	3	2.52%								

4.9 Volatility-Weighted Historical Simulation CVaR (VWHS-CVaR)

The results of the paired samples test obtained by the VWHS-CVaR model are very distinguished depending on the confidence level chosen. As we can see in table 10, for all periods of estimation used at the 95% *cl* the test points that there is no statistically significant differences between the VWHS-CVaR estimates and the losses beyond the VWHS-VaR estimates. The non-significant paired differences ranged from a lower of –0.51% and an upper of 0.72%. The mean ranged from –0.20% to 0.26%. All paired correlations for these cases were also significant. The lowest correlation value found was 0.263 for 5-year period of estimation and the highest was 0.677 for 1-year period of estimation. On the other hand, at the 99% confidence level, any period of estimation used didn't give VWHS-CVaR estimates that passed the test. The estimates seem to overestimate the tail losses. These last results, however, are not relied on accepted number of observations and, therefore, are questionable.

Considering only the results that come from a reasonable number of observations (N>30), it can be noticed that the VWHS model had better performance than previous models. The model had significant results in the test on all lengths of time used for estimation and the correlations found are all significant and higher in most cases than the others. Importantly, the model is not dependent on the choice of period of estimation, which would make a risk manager more confident on the estimates if were using the technique to evaluate tail losses of the Brazilian market index.

Ultimately, among the approaches applied in this study the nonparametric approach using volatility-weighted historical simulation to estimate, at a chosen confidence level, the maximum loss one can expect to lose if a tail event does not occur and what one can expect to lose if a tail event does occur, demonstrated to be the best tool to manage the daily risk when investing in the Brazilian stock market index.

Table 10: Paired samples statistics, correlations, differences, and *t* test for the pairs of losses exceeding VWHS-VaR measures and estimated VWHS-CVaR measures at the 95 and 99% confidence levels and different horizon periods of estimation (1, 2, 3, 4, and 5 years).

95% *cl*		Paired Samples Statistics			Paired Correlations		Paired Differences				Paired Test	
Pair	Period of est.	Mean	N	Std. Dev.	Correlation	Sig.	Mean	Std. Dev.	Lower (95% *cl*)	Upper(95% cl)	t	Sig. (2-tailed)
Loss >VWHS-VaR	1 year	-4.96%	95	2.56%	0.677	0.000	0.22%	2.00%	-0.18%	0.63%	1.095	0.276
VWHS-CVaR	1 year	-5.19%	95	2.41%								
Loss >VWHS-VaR	2 years	-5.07%	88	2.62%	0.670	0.000	0.19%	2.07%	-0.25%	0.63%	0.863	0.391
VWHS-CVaR	2 years	-5.27%	88	2.48%								
Loss >VWHS-VaR	3 years	-4.96%	65	2.39%	0.676	0.000	0.25%	1.89%	-0.22%	0.72%	1.064	0.291
VWHS-CVaR	3 years	-5.21%	65	2.30%								
Loss >VWHS-VaR	4 years	-4.30%	55	1.56%	0.571	0.000	0.26%	1.49%	-0.15%	0.66%	1.279	0.206
VWHS-CVaR	4 years	-4.56%	55	1.66%								
Loss >VWHS-VaR	5 years	-4.07%	63	1.22%	0.263	0.037	-0.20%	1.22%	-0.51%	0.11%	-1.298	0.199
VWHS-CVaR	5 years	-3.87%	63	0.64%								
99% *cl*												
Loss >VWHS-VaR	1 year	-7.99%	12	3.95%	0.927	0.000	-2.02%	1.54%	-2.99%	-1.04%	-4.551	0.001
VWHS-CVaR	1 year	-5.98%	12	3.26%								
Loss >VWHS-VaR	2 years	-7.99%	12	3.95%	0.927	0.000	-2.02%	1.54%	-2.99%	-1.04%	-4.551	0.001
VWHS-CVaR	2 years	-5.98%	12	3.26%								
Loss >VWHS-VaR	3 years	-7.67%	9	3.55%	0.932	0.000	-1.80%	1.37%	-2.85%	-0.74%	-3.936	0.004
VWHS-CVaR	3 years	-5.87%	9	2.83%								
Loss >VWHS-VaR	4 years	-6.18%	7	1.50%	0.449	0.312	-1.53%	1.40%	-2.83%	-0.23%	-2.885	0.028
VWHS-CVaR	4 years	-4.65%	7	1.09%								
Loss >VWHS-VaR	5 years	-5.84%	9	1.48%	0.324	0.395	-1.58%	1.45%	-2.70%	-0.47%	-3.283	0.011
VWHS-CVaR	5 years	-4.25%	9	0.83%								

5. Conclusions

This study aimed to verify empirically the applicability of parametric and non-parametric approaches to measure risk (VaR and CVaR) of the Brazilian market index (Ibovespa) traded on the Sao Paulo Stock Exchange. The period used for analysis goes from the first day of trade of 1995 to the last day of trade of 2004. Therefore, the total length of time for estimation of risk parameters and test of each model corresponded to 10 years. Parametric approaches assumed that daily returns follow a normal and a t-distribution. Non-parametric approaches are the historical simulation and the volatility-weighted historical simulation technique.

The results pointed out that considering the relative importance of old observations and changes in volatility give superior VaR estimates for the Brazilian market index. The volatility-weighted historical simulation approach had the best performance among the models in estimating the maximum loss one can expect to lose if a tail event does not occur in this market. For all horizons of estimation and confidence levels, the binomial test accepted the VaR predictions made by the model. The rates of losses exceeding the VaR measures ranged between 4.7-6.0%, at the 95% *cl*, and between 0.9-1.2%, at the 99% *cl*.

For large periods of estimation, the risk lines for the normal and historical simulation VaR estimates presented flatness, or excessive smoothness, demonstrating inefficiency in updating risk. The VaR estimates were too low at the 95% *cl*, and too high at the 99% *cl*. For these models, short periods of estimation give more accurate VaR estimates. Also, the use of a *t*-distribution overestimated the VaR measures. For all rejected cases, the number of losses exceeding T-VaR measures was lower than predicted. Nevertheless, for 1, 2, and 3-year periods of estimation (at the 99% *cl*), the model corrected the underestimation provided by the normal VaR.

In the task of measuring what one can expect to lose if a tail event does occur (CVaR), the volatility-weighted historical simulation approach also attained the best performance. For all periods of estimation used (1, 2, 3, 4, and 5 years), at the 95% *cl*, the samples differences paired test pointed no statistically significant differences between the VWHS-CVaR estimates and the losses beyond the VWHS-VaR estimates. All paired correlations for these cases were also significant and, in most cases, higher than found for the other models.

Inversely to the results found for the N-VaR, the parametric approach assuming normal returns provided good CVaR estimates, at the 95% confidence level, when 4 and 5-year periods of estimation were used. However, when the means were statistically different, the paired differences mean was negative in all cases, indicating that the model overestimated the value of the tail losses. For the T-CVaR measures the only chosen period of estimation and confidence level that passed the paired sample differences test was the 3- year period, at the 99% confidence level. For most of the other cases, the *t*-distribution assumption caused greater overestimation than the N-CVaR of the losses beyond its VaR. Finally, the HS-CVaR had similar performance of HS-VaR providing, at the 95% *cl*, good estimates of tail losses when short periods of estimation were used.

As a suggestion for future studies, it would be interesting to test empirically if additional refinements on the volatility forecast, for instance the use of a GARCH model, promote improvements in the risk measures provided by historical simulation approaches that use expected volatility to update return information. Additionally, as suggested by Hull and White (1998b), transformations on the return distributions that approximate them to the assumptions assumed in the parametric approaches could be used as an alternative to improve the empirical results found for these models.

6. References

Artzner, P., Delbaen, F., Ebert, J.-M., Heath, D., Thinking coherently. *Risk*, 10: 68-71, 1997.

Artzner, Ph., Delbaen, F., Ebert, J.-M., Heath, D., Coherent measures of risk, *Mathematical Finance*, 9: 203-228, 1999.

Basak, S., Shapiro, A., Value-at-risk based risk management: Optimal policies and asset prices. *The Review of Financial Studies*, 14: 371-405, 2001.

Beder, T. S., VAR: Deductive but dangerous. *Financial Analysts Journal*, 5: 12-24, 1995.

Black, F., Capital market equilibrium with restricted borrowing. *Journal of Business*, 45: 444-455, 1972.

Boudoukh, J., Richardson, M., Whitelaw, R., The best of both worlds: a hybrid approach to calculating value at risk, *Risk*, 11: 64-67, 1998.

Campbell, J. Y., Lo, A. W., MacKinlay, A. G., *The econometrics of financial markets*. Princeton, 1997.

Chen, N., Roll, R., Ross, S., Economic forces and the stock market. *Journal of Business*, 59: 383-403, 1986.

Crouhy, M., Galai, D., Mark, R., *Risk management*, McGraw-Hill, 2001.

Dowd, K., *Measuring market risk*, Wiley, 2002.

Duffie, D., Pan, J., An overview of the value at risk, *Journal of Derivatives*, 4: 7-49, 1997.

Embrechts, P., Resnick, S., Samorodnitsky, G., Living on the edge. *Risk*, 11: 96-100, 1998.

Figlewski, S., Forecasting volatility using historical data. New York University, *Working Paper*, S-94-13, (1994). In: Dowd, K., *Measuring market risk*, Wiley, 2002.

Frittelli, M., Gianin, E. R., Putting order in risk measures, *Journal of Banking & Finance*, 26: 1473-1486, 2002.

Gallati, R. R., *Risk management and capital adequacy*, McGraw-Hill, 2003.

Hawkins, I., Choosing appropriate VaR model parameters and risk-measurement methods, In: *The Professional's Handbook of Financial Risk Management*, Butterworth-Heinemann, 2000.

Hendricks, D., Evaluation of value-at-risk models using historical data, *Economic Policy Review*, 2: 39-70, 1996.

Ho, T., Key rate durations: measures of interest rate risk, *Journal of Fixed Income*, Sep, 1992.

Hull, J., White, A., Incorporating volatility updating into the historical simulation method for value-at-risk. *Journal of Risk*, 1:5-19, 1998a.

Hull, J., White, A., Value at risk when daily changes in market variables are not normally distributed. *Journal of Derivatives*, 5: 9-19, 1998b.

Jorion, P., *Value at risk: the new benchmark for managing financial risk*. McGraw-Hill, 2 ed., 2001.

JP Morgan, *RiskMetrics*, New York, 1994. In: <http://www.gloriamundi.org/> Sept/2005.

JP Morgan, *CreditMetrics*, New York, 1997. In: <http://www.gloriamundi.org/> Sept/2005.

Kupiec, P., Techniques for verifying the accuracy of risk management models, *Journal of Derivatives*, 3:73-84, 1995.

Lam, J., *Enterprise Risk Management: From Incentives to Controls*. John Wiley & Sons, Inc., 2003.

Leal, R.P.C., Rêgo, R.B., Impacto do anexo IV no mercado de capitais brasileiro. In: Mercado de Capitais – Análise Empírica no Brasil – São Paulo : Atlas, 56-66, 2000.

Lintner, J., The valuation of risky assets and the selection of risky investments in stock portfolios and capital budgets. *Review of Economics and Statistics*, 47: 13-37, 1965.

Markowitz, H. M., Portfolio selection. *Journal of Finance*, 7: 77-91, 1952.

Merton, R.C., An intertemporal capital asset pricing model. *Econometrica*, 41: 867-887, 1973.

Mossin, J., Equilibrium in a capital asset market. *Econometrica*, 34: 768-783, 1966.

Penza, P., Bansal, V. K., *Measuring market risk with value at risk.* John Wiley & Sons, Inc., 2001.

Phelan, M. J., Probability and statistics applied to the practice of financial risk management: the case of JP Morgan's Riskmetrics. *Journal of Financial Services Research*, 12:175-200, 1997.

Phoa, W., Yield curve risk factors: domestic and global contexts. In: *The Professional's Handbook of Financial Risk Management*, Butterworth-Heinemann, 2000.

Pritsker, M., *The hidden dangerous of historical simulation.* Mimeo. Board of Governors of the federal Reserve System. April, 2001. In: Dowd, K., *Measuring market risk*, Wiley, 2001.

Rockafellar, R. T., Uryasev, S., Optimization of conditional value-at-risk. *Journal of Risk*, 2: 21-41, 2000.

Rockafellar, R. T., Uryasev, S., Conditional value-at-risk for general loss distributions. *Journal of Banking & Finance*, 26: 1443-1471, 2002.

Ross, S. A., The arbitrage theory of capital asset pricing. *Journal of Economic Theory,* 13: 341-360, 1976.

Sharpe, W.F., Capital asset prices: a theory of market equilibrium under conditions of risk. *Journal of Finance*, 19: 425-442, 1964.

Szegö, G., Measures of rik. *Journal of Banking & Finance*, 26: 1253-1272, 2002.

Wilson, T. C., Infinite wisdom, *Risk*, 6:37-45, 1993.

Yamai, Y., Yoshiba, T., Value-at-risk versus expected shortfall: a practical perspective. *Journal of Banking & Finance*, 29: 997-1015, 2005.

Yamai, Y., Yoshiba, T., On the validity of value-at-risk: comparative analysis with expected shortfall. *Monetary and Economic Studies*, 20: 57-86, 2002.

Appendix A: Derivation of the EWMA model

Using the most recent *m* observations of daily returns, an unbiased estimate of the variance rate per day, σ_t^2, is:

$$\sigma_t^2 = \frac{1}{n-1}\sum_{i=1}^{n}(r_{t-i} - \bar{r})^2 \tag{29}$$

Where $\bar{r}$ is the mean of daily returns.

When dealing with daily returns the mean will be very low, and we can approximate it as been zero. Also, if n is a large number, we can replace the $n-1$ term by n. In fact, according to Figlewsky (1994), these simplifications often make very little difference to the variance estimates, and usually reduce their standard errors. We then have the following equation for the volatility estimate:

$$\sigma_t^2 = \frac{1}{n}\sum_{i=1}^{n} r_{t-i}^2 \tag{30}$$

It is easy to note that the equation above gives equal weight to all r_i^2's. However, given that the objective is to monitor the current level of volatility, it is more plausible to give greater weight to more recent observations and less weight to more distant ones. For that we can use a moving average scheme with declining weights. This approach fits better the fact that volatility tends to change over time. So, the estimate becomes:

$$\sigma_t^2 = \sum_{i=1}^{n} \alpha_i r_{t-i}^2 \tag{31}$$

where the variable α_i is the amount of weight given to the observation i days ago. It declines as i gets larger, $\alpha_i < \alpha_j$ when $i > j$, and sum to 1.

One way to deal with the fact that volatilities vary over time is to use the exponentially weighted moving average (EWMA) model. In this approach the weights, α_i, decrease exponentially as we move back trough time. Specifically, $\alpha_{i+1} = \lambda\alpha_i$, where λ is a constant between zero and one. The formula for updating volatility becomes:

$$\sigma_t^2 = (1-\lambda)\sum_{i=1}^{n} \lambda^{i-1} r_{t-i}^2 + \lambda^n \sigma_0^2 \tag{32}$$

For a large *n*, the term $\lambda^n \sigma_0^2$ is sufficiently small to be ignored. Lagging the equation above by one period, and multiplying throughout by λ, we get:

$$\lambda\sigma_{t-1}^2 \approx \lambda(1-\lambda)\sum_{i=1}^{n} \lambda^{i-1} r_{t-i-1}^2 \tag{33}$$

and

$$\lambda\sigma_{t-1}^2 = (1-\lambda)\sum_{i=1}^{n} \lambda^{i} r_{t-i-1}^2 \tag{34}$$

Subtracting equation (34) from equation (32) and rearranging gives:

$$\sigma_t^2 = \lambda\sigma_{t-1}^2 + (1-\lambda) r_{t-1}^2 - (1-\lambda)\lambda^n r_{t-n-1}^2 \tag{35}$$

Finally, the estimates of the volatility using the EWMA model becomes:

$$\sigma_t^2 \approx \lambda\sigma_{t-1}^2 + (1-\lambda) r_{t-1}^2 \tag{36}$$

Where the estimate, σ_t, of the volatility for day *t* (made at the end of day *t-1*) is obtained from σ_{t-1} (the estimate from one day ago of the volatility for day *t-1*) and r_{t-1} (the most recent observation on changes in the daily return).

Appendix B: Bovespa index profile

The Bovespa Index is a total return index weighted by traded volume and is comprised of the most liquid stocks traded on the Sao Paulo Stock Exchange. Stocks composing the index are shown in the table below.

Table 11: Stocks composing the Bovespa Index in September of 2005.

COMPANY	SIMBOL	COMPANY	SIMBOL
ACESITA-PREF	ACES4	EMBRATEL-PREF	EBTP4
AMBEV-PREF	AMBV4	GERDAU MET-PREF	GOAU4
ARACRUZ CELU-PRB	ARCZ6	GERDAU-PREF	GGBR4
BANCO ITAU-PREF	ITAU4	IPIRANGA PETR-PR	PTIP4
BELGO MINEIR	BELG3	ITAUSA-PREF	ITSA4
BRADESCO SA-PREF	BBDC4	KLABIN SA-PREF	KLBN4
BRADESPAR SA -PR	BRAP4	LIGHT	LIGH3
BRASIL	BBAS3	NET SERVICOS-PRF	NETC4
BRASIL TELE P-PR	BRTP4	PETROBRAS	PETR3
BRASIL TELE PART	BRTP3	PETROBRAS-PREF	PETR4
BRASIL TELECOM	BRTO4	SABESP	SBSP3
BRASKEM SA	BRKM5	SADIA-PREF	SDIA4
CAEMI - PREF	CMET4	SID NACIONAL	CSNA3
CELESC-PREF B	CLSC6	SOUZA CRUZ	CRUZ3
CELULAR CRT-PF A	CRTP5	TEL CTR OES-PREF	TCOC4
CEMIG	CMIG3	TELE LESTE CL-PR	TLCP4
CEMIG SA-PREF	CMIG4	TELE NORTE L-PRF	TNLP4
CESP-PREF	CESP4	TELE NORTE LESTE	TNLP3
CIA DE TRANS-PF	TRPL4	TELEMAR N L-PR A	TMAR5
COMGAS-PREF A	CGAS5	TELEMIG CELULA-P	TMCP4
CONTAX PART	CTAX3	TELESP CELUL P-P	TSPP4
CONTAX PART-PR	CTAX4	TELESP-PREF	TLPP4
COPEL-PREF B	CPLE6	TIM PART	TCSL4
ELETROBRAS	ELET3	TIM PART	TCSL3
ELETROBRAS-PR B	ELET6	UNIBANCO-UNITS	UBBR11
ELETROPAULO-PREF	ELPL4	USIMINAS SA-PF A	USIM5
EMBRAER	EMBR3	VALE R DOCE	VALE3
EMBRAER-PREF	EMBR4	VALE R DOCE-PF A	VALE5
EMBRATEL-PREF	EBTP4	VOTORANTIM-PREF	VCPA4